

FINITE HORIZON DECISION TIMING WITH PARTIALLY OBSERVABLE POISSON PROCESSES

MICHAEL LUDKOVSKI AND SEMIH O. SEZER

ABSTRACT. We study decision timing problems on finite horizon with Poissonian information arrivals. In our model, a decision maker wishes to optimally time her action in order to maximize her expected reward. The reward depends on an unobservable Markovian environment, and information about the environment is collected through a (compound) Poisson observation process. Examples of such systems arise in investment timing, reliability theory, Bayesian regime detection and technology adoption models. We solve the problem by studying an optimal stopping problem for a piecewise-deterministic process which gives the posterior likelihoods of the unobservable environment. Our method lends itself to simple numerical implementation and we present several illustrative numerical examples.

1. INTRODUCTION

Decision timing under uncertainty is one of the fundamental problems in Operations Research. In a typical setting, an economic agent (called the decision-maker or DM) has a set of possible actions \mathcal{A} where each action has a (random) reward associated with it. The objective of the DM is to select a single action and time it so as to maximize her expected reward. More precisely, the DM picks a stopping time τ and action k from the set \mathcal{A} at τ . The reward H that DM receives is a function of the pair (τ, k) , as well as of some stochastic state variable Y . In classical examples (e.g. investment timing, American option pricing, natural resource management, etc.), Y is an *observable* stochastic process (e.g. asset prices, market demand etc.), and the DM's objective is a standard optimal stopping problem.

More complicated stopping problems involving *unobserved* system states have also been considered in the literature; see, for example, [2], [21], [31], [30], [24], [38], [34], [18], [13], [11]. Such models are especially natural when one wishes to capture the inherent conflict between gathering of information (which makes waiting valuable) and the time-value of money (which makes waiting costly). Indeed, most realistic settings involve a DM who is only partially aware of the environment and must collect data before making a decision. In a multi-period setting, it is natural to capture this uncertainty in the environment through an unobservable stochastic process $M \equiv \{M_t\}_{t \geq 0}$, where M_t represents the state of the world at time t . The DM starts with an initial guess about M , collects information via relevant news, and updates her beliefs. At the time of decision she then receives a reward that depends on the present environment, $H = H(\tau, k, M_\tau)$.

In such problems, a common approach is to postulate that the process M is a partially observable Markov (decision) process (POMDP), in which case we have a hidden Markov model (HMM). We refer the reader to [5], [14] for a comprehensive treatment of discrete-time models and to [4], [27] for continuous-time models and applications.

In both discrete- and continuous-time models the analysis separates the sub-problems of estimation (filtering of M) and control. The second “control” step requires re-formulating the problem under an equivalent fully observable system, where the conditional distributions/probabilities of

2000 *Mathematics Subject Classification.* Primary 62L10; Secondary 62L15, 62C10, 60G40.

Key words and phrases. Markov-modulated Poisson processes, Bayesian sequential analysis, optimal stopping, decision making.

the process M constitute the new state variables. In discrete-time, the value function is typically a fixed point of the corresponding dynamic programming (DP) operator, and can be obtained via a recursive application of this operator; see, for example, the models and algorithms in [5], [28]. On the other hand, continuous-time formulations allow more sophisticated models, and the dynamic programming principle generally manifests itself in the form of a (partial) differential (delay) equation; see [17], [26], [4], [33, Chapter 6] and the references therein for various examples.

The major distinction between discrete- and continuous-time models comes from the nature of the control and the observations; that is, is the system *asynchronous* and observations/stopping can occur anytime, or are there fixed time epochs when new information is processed and stopping decisions are made. A similar distinction exists within continuous-time models. If news (such as changes in asset prices) arrive in infinitesimal amounts, then it is intuitive to have a continuum of information, which is typically captured by the filtration of an observed diffusion process. However, in many instances, a more realistic representation is to use “discrete” information amounts. Corporate developments, engineering failures, insurance claims, and economic surveys are all discrete events and the corresponding news arrive in “chunks”. Note that discreteness of information is distinct from the discreteness of time. The model is still in continuous-time, since the events may take place at any instance. However, the event itself carries a strictly positive amount of information. Moreover, “no news” is still informative and affects the beliefs of the DM.

Mathematically, discrete information in continuous-time may be represented by the filtration of an observed marked point process. In such a model, the instantaneous arrival intensity and the distribution of the marks of the point process typically depend on the current state of the process M . That is, the observable point process encodes information about the hidden environment M via its arrival times and/or marks. Filtering with continuous-time point process observations has been considered in [6, 1, 15], and it is known that the dynamics of the conditional probabilities of M are of the piecewise deterministic process (PDP) type. In other words, the DM beliefs evolve *deterministically* between arrivals of new information, and experience random jumps at event times. From the control perspective, various aspects of optimal stopping of PDP’s have been studied by [26], [20] and [7].

In this paper, we study a class of finite-horizon decision-making problems within the PDP framework by considering a general regime-switching model with Poisson information arrivals. Poissonian information allows us to capture the discreteness of news while maintaining a rich framework for the dependence of the observable X on the unobservable state of M , which can manifest itself both in arrival rate and mark distribution effects. In this context, our main contribution is the full characterization of the value function and optimal policy of the DM, with a direct proof of the dynamic programming principle and characterization of the optimal and ϵ -optimal policies. Our approach also yields a numerical algorithm that can be readily implemented (see Section 6 for examples). Within the PDP framework, related problems have been considered by [24] in connection with system reliability studies, [23] and [34] in the context of insurance premium re-pricing and [32], [19], [3], [12] for classical Poisson disorder and regime detection problems.

Our model provides a non-trivial generalization of previous analysis of decision making under Poissonian information structures. More precisely, we extend existing literature in three directions. First, we consider a general continuous-time finite-state Markov chain for the environment variable M (without any assumptions on the transition rates), and impose no restriction on the arrival rate and mark distribution of the observed compound Poisson process X . The latter allows us to model any setting where the DM also gets information via the *size/type* of each event besides the interarrival epochs. Second, we consider a general discount/cost structure, that can be used to encode a variety of economic objectives. Finally, we work in the context of finite horizon, where value functions are time-inhomogeneous. This is a more realistic setting since a practicing

DM typically has a well-defined “window” for making their decision. The introduction of time-to-maturity as a state variable makes the numerical computation more challenging and leads to appearance of new effects that are not possible with stationary models. At the same time, our model allows a natural interpolation from finite to infinite horizon; see Section 4.4.

Before concluding our discussion here, let us mention that the choice of “discrete-time model” versus “continuous-time model with discrete information” will be made according to the preferences of the modeler, as well as the nature of the problem. Accordingly, similar applications may invite different modeling approaches; for instance, the machine reliability problem discussed in Section 1.1 below was studied both in a discrete-time setting by [37], a continuous-time setting by [24] and even a hybrid continuous-time model with discrete-epoch observations in [29]. In this context, if the machine/production system is subject to major breakdowns, then continuous monitoring may be more desirable. In other cases, end-of-day inspections may be more than enough to restore the profitability of operations. While the aforementioned formulations are superficially similar (and in some specific cases even equivalent, see [16]), the respective solution methods utilize quite different tools. The solution of discrete time models generally relies on the Smallwood-Sondik property [36] that shows that with finite state, observation, and action spaces the value function is piecewise linear and convex. In continuous-time this property no longer holds, and the smoothness of the value function must be independently established. Also in discrete-time models decisions and controls are intrinsically paired with observations. In contrast, in the models considered here, the control may take place both at event time or between events, which is an important qualitative distinction.

1.1. A catalogue of sample problems. Since the framework studied throughout the paper is general, let us first provide a number of motivating examples illustrating the applications in various settings.

Profit Maximization with Information Cost. Let us consider an insurance company which is planning to launch a new policy/product to its clients. The frequency of corresponding insurance claims and the severity of claim sizes are not known precisely. Rather, they depend on the current quality of the insurance portfolio, represented by a Markov process $M = \{M_t\}_{t \geq 0}$ taking values on some space $E \triangleq \{1, \dots, n\}$. Once the policy is launched, it yields a random payoff that depends on the current state of M only. To model this, we say that when M is at state $i \in E$ at the launch-time, the random payoff is given by an independent random variable Φ_i with some finite mean $\mu_i = \mathbb{E}[\Phi_i]$.

Information about M is obtained through the filed claims process $X = \{X_t\}_{t \geq 0}$ received by the firm. The cumulative claim process has the form $X_t = \sum_{j=1}^{N_t} Y_j$ for $t \geq 0$. Here N_t is the total number of claims up to time t , and Y_j is the size for the j 'th claim for $j \in \mathbb{N}$. The process N is a simple Poisson process with intensity λ_i whenever M is at state $i \in E$. Moreover, if a claim is known to occur when M is at state i , the claim size is an independent random variable with distribution ν_i .

At any time prior to some terminal time $T < \infty$, the company may launch the product or permanently abandon it. Alternatively, it can delay this decision to obtain more information on M , and to increase the likelihood of catching M at a favorable state. However, waiting for additional information costs $c \leq 0$ per unit time. Therefore, the company must decide how long it observes X prior to a decision, and what decision (launch vs. quit) should be taken at that time.

Let $\tau \leq T$ denote the decision time, and let the random variable $d \in \{0, 1\}$ indicate whether the product is released or abandoned. That is, on the event $\{d = 1\}$ the company launches the product, and on $\{d = 0\}$ it quits. Clearly, the time τ should be determined based on the observations from the claim process X , and the choice of action d should be determined solely by the information

generated by X until τ . Then, the objective of the company is to compute

$$(1.1) \quad \sup_{\tau, d} \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} c dt + e^{-\rho \tau} 1_{\{d=1\}} \left(\sum_{i \in E} \mu_i \cdot 1_{\{M_\tau=i\}} \right) \right]$$

over all such pairs (τ, d) . In (1.1), $\rho > 0$ is a given discount rate used by the company in reference to future revenues, and $\vec{\pi} \equiv (\pi_1, \dots, \pi_n) \triangleq (\mathbb{P}(M_0 = 1), \dots, \mathbb{P}(M_0 = n))$ denote the initial beliefs of the company about the state of M at $t = 0$.

A related problem has been considered on infinite horizon by [34] who maximizes future risk reserves of the insurance company where at the time τ the company will re-calculate its premiums. We also refer the reader to [13], and [39] for recent work on timing project commitment/abandonment in continuous and discrete time respectively.

Bayesian Regime Detection. In this problem, a compound Poisson process $X = \{X_t\}_{t \geq 0}$ is observed starting from $t = 0$. The arrival rate λ and mark distribution ν of X are not known precisely. Rather they depend on the static regime of the Markov process M with n absorbing states (i.e., $M_t = M_0$ for all $t \geq 0$). Each state corresponds to the realization of one of the n simple hypotheses

$$(1.2) \quad H_1 : (\lambda, \nu) = (\lambda_1, \nu_1), \quad \dots, \quad H_n : (\lambda, \nu) = (\lambda_n, \nu_n),$$

with given prior likelihoods π_i , for $i = 1, \dots, n$. The objective of the DM is to recognize the current regime as quickly as possible, with minimal probability of wrong decision.

In earlier work on this problem, the trade-off between observing and stopping is generally modeled via the Bayes risk

$$(1.3) \quad \mathbb{E}^{\vec{\pi}} \left[\tau + \sum_{k,i=1}^n \mu_{k,i} 1_{\{d=k, M_0=i\}} \right],$$

where τ is the decision time, $d \in \{1, \dots, n\}$ represents the hypothesis selected and $\mu_{k,i} \geq 0$ is the cost of selecting the wrong hypothesis H_k when the correct one is H_i . The DM then needs to minimize (1.3) and find a pair (τ, d) , if one exists, that attains this infimum.

The infinite horizon version of (1.3) was solved for the first time by [32] for a simple Poisson process with $n = 2$. Later, [19] provided the solution (again with $n = 2$), where the jump size is exponentially distributed under each hypothesis, with the mean of the exponential distribution the same as the proposed arrival rate. The solution for any jump distribution and for $n \in \mathbb{N}$ was recently provided by [12]. Our model in this paper can be viewed as the finite horizon version of that problem, where a decision must be made before a terminal time $T < \infty$.

Optimal Replacement Time of a Reliability System. [24] consider an optimal stopping problem in reliability with a partially observed Poisson process. The problem is to find when to discard or replace a machine/production-system whose production quality deteriorates over time due to the usual wear-and-tear. The status of the machine is modeled with a finite state Markov process M . The process moves from good states to bad states over time. Eventually it ends in the n 'th absorbing state which represents an unacceptable quality level.

The DM observes the failure times $\sigma_1, \sigma_2, \dots$ (the failures can also be interpreted as defective items in the context of a machine); it is assumed that the corresponding ‘‘arrivals’’ form a Poisson process whose intensity is λ_i when the current state of the process M is $i \in E = \{1, \dots, n\}$. Running the system in state i yields a net payoff $c_i \in \mathbb{R}$ per unit time. A high c_i indicates that the machine is profitable, while a negative c_i , including the assumed $c_n < 0$, means that the low quality outweighs the benefits. At any time the DM can stop running the machine and replace it,

with a terminal cost of μ_i if the process M happens to be in state $i \in E$ at that time. [24] then solve the problem of maximizing

$$(1.4) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau \sum_{i \in E} c_i 1_{\{M_t=i\}} dt + \sum_{i \in E} \mu_i \cdot 1_{\{M_\tau=i\}} \right],$$

over all random time τ 's (whose value is determined by the history generated by the arrival process) and under certain assumptions on the arrival rates λ_i 's, the infinitesimal generator of M , and cost parameters c_i, μ_i 's. Related models have appeared in [29], and [37] and go all the way to classical POMDP work by [36]. In this paper, we consider that problem without any parameter assumptions and with the additional finite horizon constraint $\tau \leq T$.

1.2. Problem description: a unifying framework. In the examples above, a DM observes a compound Poisson process X with arrival rate λ , and mark/jump distribution ν . The local characteristics (λ, ν) of X are determined by the current state of an *unobservable* finite-state Markov process M .

At any time τ less than some $T < \infty$, the DM can stop and select an action k from the set $\mathcal{A} \triangleq \{1, \dots, a\}$. If action $k \in \mathcal{A}$ is taken, this yields a terminal reward/payoff of

$$\sum_{i \in E} \mu_{k,i} \cdot 1_{\{M_\tau=i\}}$$

as a function of the unobservable state of M . Here, $\mu_{k,i}$ is a given finite (not necessarily positive) number. One can also interpret $\mu_{k,i}$ as the expected value of an independent random variable $\Phi_{k,i}$ representing the uncertain payoff of taking action k when $M_t = i$. Also note that if there is a time-lag between the decision and its realization, and if this delay is independent, then $\mu_{k,i}$ can be assumed to be the expected discounted value of this payoff.

The DM may alternatively delay her decision and continue to observe the process X in order to collect more information, or in order to stop later when M appears to be in a better state. Delaying the decision carries associated costs (rewards) due to the cost of observation or lost opportunity (or operating revenues). We allow these terms to depend on M and we assume that an amount with present value

$$\int_0^\tau e^{-\rho t} \left(\sum_{i \in E} c_i 1_{\{M_t=i\}} \right) dt$$

is accumulated until the decision time τ . Here $\rho \geq 0$ is the discount factor, and c_i is the instantaneous cost or revenue of running the system when M is at state $i \in E$. We allow ρ to be zero. This makes the formulation suitable for non-financial application where the quality of the decision is more important than its timing.

In this setup, the objective of the DM is to find an *admissible* strategy that will maximize her total expected reward and resolve the trade-off between *exploring* (getting more observations) and *exploiting* (engaging in an action). An admissible strategy is a pair (τ, d) , where $\tau \leq T$ is the decision time and $d \in \mathcal{A}$ is the action selected at this time. Since the DM collects information from observing X , the value of τ should be determined by the information generated by X , namely τ must be a stopping time of the filtration \mathcal{F}^X of X . Also, the decision variable d should be measurable with respect to the information \mathcal{F}_τ^X revealed by X until τ . Let $\vec{\pi} = (\pi_1, \dots, \pi_n) \triangleq (\mathbb{P}(M_0 = 1), \dots, \mathbb{P}(M_0 = n))$ be the initial (prior) beliefs of the DM about M and $\mathbb{P}^{\vec{\pi}}$ the corresponding conditional probability

law. Then the objective of the DM is to compute

(1.5)

$$U(T, \vec{\pi}) \triangleq \sup_{\tau \leq T, d \in \mathcal{F}_T^X} \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} \left(\sum_{i \in E} c_i 1_{\{M_t=i\}} \right) dt + e^{-\rho \tau} \sum_{k \in \mathcal{A}} 1_{\{d=k\}} \left(\sum_{i \in E} \mu_{k,i} \cdot 1_{\{M_\tau=i\}} \right) \right],$$

and, if it exists, find an admissible pair (τ, d) attaining this value.

In Section 2 below we describe the formal setting of our model and show that the problem in (1.5) is equivalent to an optimal stopping problem in terms of the *conditional probability process*, which is a piecewise deterministic process. Section 3 describes how the value function of this stopping problem can be computed via a sequential procedure. The results of Section 3 are used in Section 4 in order to identify an optimal strategy and describe its properties. Following this, Section 5 explores alternative objective functions that can be employed in our framework. Finally, in Section 6 we give numerical examples illustrating our results. Most of the proofs are delegated to the Appendices at the end.

2. PROBLEM STATEMENT

2.1. Model. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space hosting a continuous-time Markov process M taking values on $E \triangleq \{1, \dots, n\}$, for $n \in \mathbb{N}$, and with infinitesimal generator $Q = (q_{ij})_{i,j \in E}$. Also, we have a collection of independent compound Poisson processes $X^{(1)}, \dots, X^{(n)}$ with local parameters $(\lambda_1, \nu_1), \dots, (\lambda_n, \nu_n)$ respectively. In terms of these independent processes, we define the observation process

$$(2.1) \quad X_t \triangleq X_0 + \int_{(0,t]} \sum_{i \in E} 1_{\{M_s=i\}} dX_s^{(i)}, \quad t \geq 0,$$

which is a Markov-modulated Poisson process, also called a Cox process (see [8]). In the remainder, we let $\sigma_0, \sigma_1, \dots$ denote the arrival times of the process X :

$$\sigma_m \triangleq \inf\{t > \sigma_{m-1} : X_t \neq X_{t-}\}, \quad m \geq 1, \quad \text{with } \sigma_0 \equiv 0,$$

and the variables Y_1, Y_2, \dots denote \mathbb{R}^d -valued marks observed at these arrival times:

$$Y_m = X_{\sigma_m} - X_{\sigma_m-}, \quad m \geq 1.$$

Finally, to compute relative likelihoods of different marks, we introduce the total measure ν defined as $\nu \triangleq \nu_1 + \dots + \nu_n$, and we let $f_i(\cdot)$ be the density of ν_i with respect to ν .

2.2. Conditional probability process. For a point in $D \triangleq \{\vec{\pi} \in \mathbb{R}_+^n : \pi_1 + \dots + \pi_n = 1\}$, let $\mathbb{P}^{\vec{\pi}}$ denote the probability measure (with the expectation operator $\mathbb{E}^{\vec{\pi}}$) under which M has initial distribution $\vec{\pi}$. Moreover, let $\mathbb{F} \triangleq \{\mathcal{F}_t^X\}_{t \geq 0}$ be the filtration of the process X in (2.1). With this notation, we define the D -valued *conditional probability process* $\vec{\Pi}_t \triangleq (\Pi_t^{(1)}, \dots, \Pi_t^{(n)})$ such that

$$(2.2) \quad \Pi_t^{(i)} = \mathbb{P}^{\vec{\pi}}\{M_t = i | \mathcal{F}_t^X\}, \quad \text{for } i \in E, \text{ and } t \geq 0.$$

The process $\vec{\Pi}$ is clearly adapted to \mathbb{F} , and each component gives the conditional probability that the current state of M is $\{i\}$ given the information generated by X until the current time t . Moreover, using standard arguments as in [35, pp. 166-167], and [12, Proof of Proposition 2.1], it can be shown that the problem in (1.5) is equivalent to a fully observed optimal stopping problem with the process $\vec{\Pi}$ as the new hyperstate. More precisely, the value function U in (1.5) can be written as

$$(2.3) \quad U(T, \vec{\pi}) = V(T, \vec{\pi}) \triangleq \sup_{\tau \leq T} \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right],$$

in terms of the functions

$$(2.4) \quad C(\vec{\pi}) \triangleq \sum_{i \in E} c_i \pi_i \quad \text{and} \quad H(\vec{\pi}) \triangleq \max_{k \in \mathcal{A}} H_k(\vec{\pi}), \quad \text{where} \quad H_k(\vec{\pi}) \triangleq \sum_{i \in E} \mu_{k,i} \pi_i.$$

If there is a stopping time τ^* attaining the supremum in (2.3), then the admissible strategy $(\tau^*, d(\tau^*))$ is an optimal rule for the problem in (1.5) if we define

$$(2.5) \quad d(\tau) \in \arg \max_{k \in \mathcal{A}} H_k(\vec{\Pi}_\tau).$$

2.3. Sample paths of $\vec{\Pi}$. Let us take a sample path of the observations process X , in which m -many arrivals are observed on $[0, t]$. Let $(t_k)_{k \leq m}$ denote those arrival times. If we know that the process M stays at the state $\{i\}$ without any transition, then the (conditional) likelihood of this path would be written as $\mathbb{P}^{\vec{\pi}}\{\sigma_k \in dt_k, Y_k \in dy_k; k \leq m \mid M_s = i, s \leq t\} =$

$$[\lambda_i e^{-\lambda_i t_1} dt_1] \cdots [\lambda_i e^{-\lambda_i(t_m - t_{m-1})} dt_m] e^{-\lambda_i(t_m - t_{m-1})} \prod_{k=1}^m [f_i(y_k) \nu(dy_k)] = e^{-\lambda_i t} \prod_{k=1}^m \lambda_i dt_k \cdot f_i(y_k) \nu(dy_k).$$

By construction, the observation process X has independent increments conditioned on $M = \{M_t\}_{t \geq 0}$. Therefore, we have

$$(2.6) \quad 1_{\{M_t=i\}} \cdot \mathbb{P}^{\vec{\pi}}\left\{\sigma_i \in dt_i, Y_i \in dy_i; i \leq m \mid M_s; s \leq t\right\} \\ = 1_{\{M_t=i\}} \cdot \exp\left(-\int_0^t \sum_{i=1}^n \lambda_i 1_{\{M_{t_k}=i\}} ds\right) \cdot \prod_{k=1}^m \left(\sum_{j \in E} 1_{\{M_{t_k}=j\}} [\lambda_j dt_k \cdot f_j(y_k) \nu(dy_k)]\right).$$

By taking the expectations of the expressions above, we obtain the unconditional likelihoods, in terms of which we give an explicit representation for the process $\vec{\Pi}$ in Lemma 2.1 below.

Lemma 2.1. *For $i \in E$, let us define*

$$(2.7) \quad L_i^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m) \triangleq \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} \cdot e^{-I(t)} \cdot \prod_{k=1}^m \ell(t_k, y_k) \right],$$

where

$$(2.8) \quad I(t) \triangleq \int_0^t \sum_{i=1}^n \lambda_i 1_{\{M_s=i\}} ds \quad \text{and} \quad \ell(t, y) \triangleq \sum_{j \in E} 1_{\{M_t=j\}} \lambda_j \cdot f_j(y).$$

Also, let $L^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m) \triangleq \sum_{j \in E} L_j^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m)$. Then we have

$$(2.9) \quad \Pi_t^{(i)} = \frac{L_i^{\vec{\pi}}(t, N_t : (\sigma_k, Y_k), k \leq N_t)}{L^{\vec{\pi}}(t, N_t : (\sigma_k, Y_k), k \leq N_t)} \equiv \left[\frac{L_i^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m)}{L^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m)} \right] \Bigg|_{m=N_t; (t_k=\sigma_k, y_k=Y_k)_{k \leq m}},$$

$\mathbb{P}^{\vec{\pi}}$ -a.s., for all $t \geq 0$, and for $i \in E$.

Lemma 2.1 indicates that the conditional probability of M_t being in state i is simply the (unconditional) relative likelihood of the observed path until t on the event $\{M_t = i\}$. Using the explicit form in (2.9), we describe the behavior of the sample paths of $\vec{\Pi}$ in Remark 2.1 below.

Remark 2.1. *The process $\vec{\Pi}$ has piecewise-deterministic sample paths: between two arrival times of X , it moves deterministically, and at an arrival time, it jumps from one point to another depending on the observed mark size (see Figure 2.3). In precise terms, the sample paths have the*

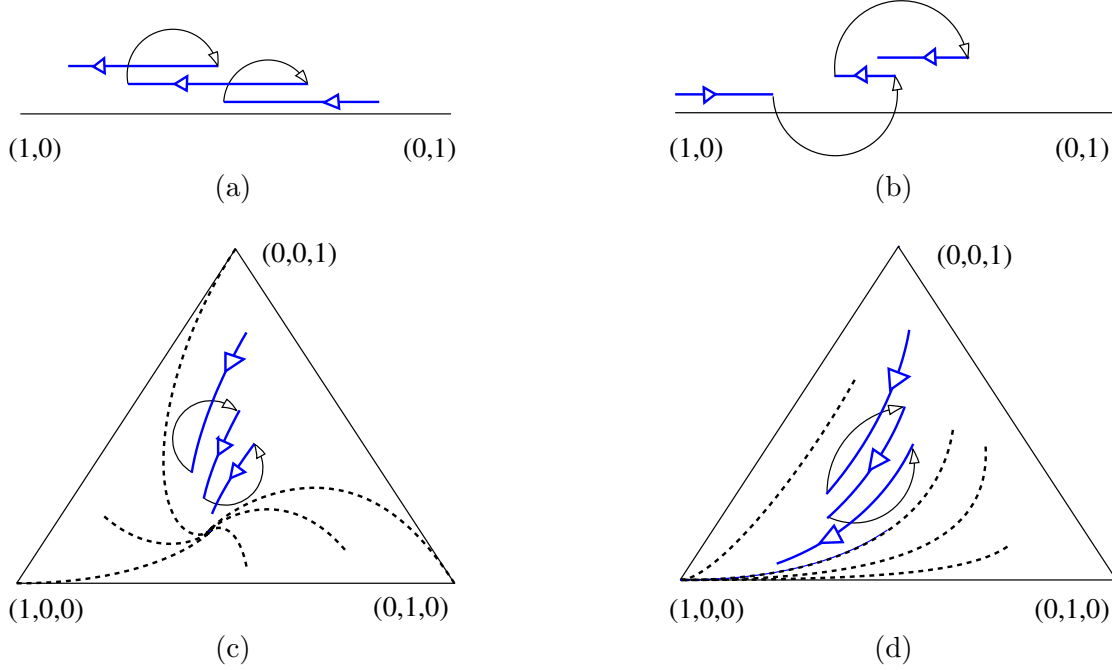


FIGURE 1. Sample paths of the process $\vec{\Pi}$ for different examples. Solid lines represent actual sample paths. Dashed lines in panels (c) and (d) are the deterministic parts in (2.11). In panels (a) and (b), there are two hidden states, and in panels (c) and (d), there are three. In each example, jumps of the process X are always of unit size. The parameters of each example:

$$Q_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_b = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q_c = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad Q_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\vec{\lambda}_a = [1, 2]$, $\vec{\lambda}_b = [1, 4]$, $\vec{\lambda}_c = [1, 2, 3]$, $\vec{\lambda}_d = [1, 3, 5]$.

characterization

$$(2.10) \quad \left\{ \begin{array}{l} \vec{\Pi}(t) = \vec{x}(t - \sigma_m, \vec{\Pi}(\sigma_m)), \quad \sigma_m \leq t < \sigma_{m+1}, \quad m \in \mathbb{N} \\ \vec{\Pi}(\sigma_m) = \left(\frac{\lambda_1 f_1(Y_m) \Pi_1(\sigma_m -)}{\sum_{j \in E} \lambda_j f_j(Y_m) \Pi_j(\sigma_m -)}, \dots, \frac{\lambda_n f_n(Y_m) \Pi_n(\sigma_m -)}{\sum_{j \in E} \lambda_j f_j(Y_m) \Pi_j(\sigma_m -)} \right) \end{array} \right\},$$

where $\vec{x}(t, \vec{\pi}) \equiv (x_1(t, \vec{\pi}), \dots, x_n(t, \vec{\pi}))$ is defined as

$$(2.11) \quad x_i(t, \vec{\pi}) \triangleq \frac{\mathbb{P}^{\vec{\pi}}\{\sigma_1 > t, M_t = i\}}{\mathbb{P}^{\vec{\pi}}\{\sigma_1 > t\}} = \frac{\mathbb{E}^{\vec{\pi}}[1_{\{M_t=i\}} \cdot e^{-I(t)}]}{\mathbb{E}^{\vec{\pi}}[e^{-I(t)}]}, \quad \text{for } i \in E,$$

and satisfy the semigroup property $\vec{x}(t+u, \vec{\pi}) = \vec{x}(u, \vec{x}(t, \vec{\pi}))$, for $t, u \geq 0$.

The i 'th component $x_i(\cdot, \cdot)$ indicates how likely it is to have a period of $[0, t]$ without any arrival on the event $\{M_t = i\}$, as expected. Moreover, for $0 \leq u_1 \leq u_2 \leq \dots \leq u_k$ and for a bounded

function $g(\cdot)$, we have

$$\begin{aligned}
 (2.12) \quad \mathbb{E}^{\vec{\pi}} \left[g(X_{t+u_1} - X_t, \dots, X_{t+u_k} - X_t) \middle| \mathcal{F}_t^X \right] \\
 = \sum_{j \in E} \mathbb{P}\{M_t = j \mid \mathcal{F}_t^X\} \cdot \mathbb{E}^{\vec{\pi}} \left[g(X_{t+u_1} - X_t, \dots, X_{t+u_k} - X_t) \middle| \mathcal{F}_t^X, M_t = j \right] \\
 = \sum_{j \in E} \Pi_j(t) \cdot \mathbb{E} \left[g(X_{u_1}, \dots, X_{u_k}) \middle| M_0 = j \right] = \mathbb{E}^{\vec{\Pi}_t} \left[g(X_{u_1}, \dots, X_{u_k}) \right],
 \end{aligned}$$

where the first equality in the last line follows from the construction of the process X in (2.1). The equation (2.12) together with the characterization in (2.10) implies that $\vec{\Pi}$ is a $(\mathbb{P}^{\vec{\pi}}, \mathbb{F})$ -Markov process for every $\vec{\pi} \in D$.

Corollary 2.1. Using infinitesimal last step analysis, it can be shown (see, for example, [9, page 416], and [25, Chapter 6.7]) that the vector

$$(2.13) \quad \vec{m}(t, \vec{\pi}) \equiv (m_1(t, \vec{\pi}), \dots, m_n(t, \vec{\pi})) \triangleq \left(\mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=1\}} \cdot e^{-I(u)} \right], \dots, \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=n\}} \cdot e^{-I(u)} \right] \right)$$

has the form $\vec{m}(t, \vec{\pi}) = \vec{\pi} \cdot e^{t(Q-\Lambda)}$ where Λ is the $n \times n$ diagonal matrix with $\Lambda_{i,i} = \lambda_i$, and the components of $\vec{m}(t, \vec{\pi})$ solve $dm_i(t, \vec{\pi})/dt = -\lambda_i m_i(t, \vec{\pi}) + \sum_{j \in E} m_j(t, \vec{\pi}) \cdot q_{j,i}$. Then together with the chain rule and (2.11) we obtain

$$(2.14) \quad \frac{dx_i(t, \vec{\pi})}{dt} = \left(\sum_j^n q_{j,i} x_j(t, \vec{\pi}) - \lambda_i x_i(t, \vec{\pi}) + x_i(t, \vec{\pi}) \sum_j^n \lambda_j x_j(t, \vec{\pi}) \right).$$

Hence, the process $\vec{\Pi}$ in (2.10) has the dynamics

$$(2.15) \quad d\Pi_t^{(i)} = \left(\sum_j^n q_{j,i} \Pi_{t-}^{(j)} - \lambda_i \Pi_{t-}^{(i)} + \Pi_{t-}^{(i)} \sum_j^n \lambda_j \Pi_{t-}^{(j)} \right) dt + \int_{\mathbb{R}^d} \left[\frac{\lambda_i f_i(y) \Pi_{t-}^{(i)}}{\sum_{j \in E} \lambda_j f_j(y) \Pi_{t-}^{(j)}} - 1 \right] p(dt, dy), \quad i \in E,$$

where $p(\cdot, \cdot)$ is the point process generated by X ; that is

$$p((0, t] \times B) = \sum_{i \in \mathbb{N}} 1_{(0, t] \times B}(\sigma_i, Y_i), \quad \text{for every Borel set } B \in \mathcal{B}(\mathbb{R}^d) \text{ and } t \geq 0.$$

3. CONSTRUCTING THE VALUE FUNCTION

The characterization of the sample paths in (2.15) and general theory of optimal stopping (see, for example, [4, 26]) imply that the free-boundary problem associated with the optimal stopping problem in (2.3) has the form

$$(3.1) \quad \max\{(-\rho + \mathcal{L})V(s, \vec{\pi}) + C(\vec{\pi}); H(\vec{\pi}) - V(s, \vec{\pi})\} = 0,$$

in terms of the infinitesimal generator

$$\begin{aligned}
 \mathcal{L}V(s, \vec{\pi}) &= \frac{\partial V(s, \vec{\pi})}{\partial s} + \sum_{i \in E} \left(\sum_{j \in E} q_{j,i} \pi_j - \lambda_i \pi_i + \pi_i \sum_{j \in E} \lambda_j \pi_j \right) \frac{\partial V(s, \vec{\pi})}{\partial \pi_i} \\
 &\quad + \int_{y \in \mathbb{R}^d} \left[V\left(s, \frac{\lambda_1 \pi_1 f_1(y)}{\sum_{j \in E} \lambda_j \pi_j f_j(y)}, \dots, \frac{\lambda_n \pi_n f_n(y)}{\sum_{j \in E} \lambda_j \pi_j f_j(y)}\right) - V(s, \vec{\pi}) \right] \sum_{i \in E} \pi_i \lambda_i \nu_i(dy),
 \end{aligned}$$

of the process $\vec{\Pi}$. The infinitesimal generator \mathcal{L} is a partial differential-difference operator on $[0, T] \times D \subset \mathbb{R}^{n+1}$. Hence, solving the equation $(-\rho + \mathcal{L})V(s, \vec{\pi}) + C(\vec{\pi}) = 0$ and determining the

boundary of the region $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$ is not easy even when $n = 2$; see, for example, [32] who solve free-boundary problems similar to (3.1) for infinite horizon problems, and with $n = 2$.

Instead of studying the problem in (3.1), we will employ a sequential approximation technique to compute the value function following [20] and [10, Chapter 5]. Similar approach is also taken in [3] and [12] for disorder-detection and hypothesis-testing problems respectively in infinite horizon. Since our problem is in finite-horizon, we work with time-dependent operators, and this requires non-trivial modifications of their arguments. The method is described in the sequel, and the proofs are given in the Appendix.

3.1. A sequential approximation. Let us first define the functions

(3.2)

$$V(s, \vec{\pi}) \triangleq \sup_{\tau \leq s} \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right], \quad \text{and}$$

$$V_m(s, \vec{\pi}) \triangleq \sup_{\tau \leq s} \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right], \quad \text{for } m \in \mathbb{N}, \text{ on } [0, T] \times D,$$

where the first argument ‘ s ’ should be considered as the remaining time to maturity.

Proposition 3.1 below shows that V_m ’s converge to V uniformly; see also the proof of [10, Theorem (53.40)] and [12, Proposition 3.1] for related results. Proposition 3.1 is a generalization of these results in the finite horizon case.

Proposition 3.1. *The sequence $\{V_m\}_{m \geq 1}$ converges to V uniformly on $[0, T] \times D$. More precisely, we have*

$$(3.3) \quad V_m(s, \vec{\pi}) \leq V(s, \vec{\pi}) \leq V_m(s, \vec{\pi}) + (T\|C\| + 2\|H\|) \left(\frac{\bar{\lambda}T}{m-1} \right)^{1/2} \cdot \left(\frac{\bar{\lambda}}{2\rho + \bar{\lambda}} \right)^{m/2},$$

for all $(s, \vec{\pi}) \in [0, T] \times D$ and $m \in \mathbb{N}$, where $\|C\| \triangleq \max_{\vec{\pi} \in D} |C(\vec{\pi})|$, $\|H\| \triangleq \max_{\vec{\pi} \in D} |H(\vec{\pi})|$ and $\bar{\lambda} \triangleq \max_{i \in E} \lambda_i$.

Let us consider the second problem in (3.2) for fixed $m \in \mathbb{N}$, and let $\tau \leq s$ be a \mathbb{F} -stopping time. Then, the dynamic programming intuition suggests that $V(\cdot)$ should solve the equation $V_m(s, \vec{\pi}) = J_0 V_{m-1}(s, \vec{\pi})$, where the operator J_0 is defined as

(3.4)

$$J_0 w(s, \vec{\pi}) \triangleq \sup_{\tau \leq s} \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\tau < \sigma_1\}} e^{-\rho \tau} H(\vec{\Pi}_\tau) + 1_{\{\sigma_1 \leq \tau\}} e^{-\rho \sigma_1} w(s - \sigma_1, \vec{\Pi}(\sigma_1)) \right],$$

for a bounded function $w : [0, T] \times D \mapsto \mathbb{R}$.

The following characterization of \mathbb{F} -stopping times is from [6, Theorem T33, p. 308] and [10, Lemma A2.3, p. 261].

Lemma 3.1. *For every \mathbb{F} -stopping time (bounded as $\tau \leq s \leq T$), and for every $m \in \mathbb{N}$, there exists a $\mathcal{F}_{\sigma_m}^X$ -measurable random variable R_m such that $\tau \wedge \sigma_{m+1} = (\sigma_m + R_m) \wedge \sigma_{m+1}$, \mathbb{P} -almost surely on $\{\tau \geq \sigma_m\}$.*

Lemma 3.1 implies that the supremum in (3.4) can equivalently be taken over deterministic times, in which case the same problem becomes

$$(3.5) \quad V_m(s, \vec{\pi}) = J_0 V_{m-1}(s, \vec{\pi}) \triangleq \sup_{t \in [0, s]} J V_{m-1}(t, s, \vec{\pi}),$$

where the operator J has the form

$$(3.6) \quad Jw(t, s, \vec{\pi}) \triangleq \mathbb{E}^{\vec{\pi}} \left[\int_0^{t \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{t < \sigma_1\}} e^{-\rho t} H(\vec{\Pi}_t) + 1_{\{\sigma_1 \leq t\}} e^{-\rho \sigma_1} w(s - \sigma_1, \vec{\Pi}(\sigma_1)) \right].$$

Note that, with the notation in (2.13), we have

$$\mathbb{P}^{\vec{\pi}}[\sigma_1 > u] = \mathbb{E}^{\vec{\pi}}[e^{-I(u)}] \quad \text{and} \quad \mathbb{P}^{\vec{\pi}}[\sigma_1 \in du, M_u = i] = \mathbb{E}^{\vec{\pi}}[\lambda_i 1_{\{M_u = i\}} e^{-I(u)}] du = \lambda_i m_i(u, \vec{\pi}) du,$$

and using the characterization of the paths in (2.10) and (2.14) the operator J in (3.6) can be rewritten as

$$(3.7) \quad Jw(t, s, \vec{\pi}) = \mathbb{E}^{\vec{\pi}}[e^{-I(t)}] \cdot e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi})) \\ + \int_0^t e^{-\rho u} \sum_{i \in E} m_i(u, \vec{\pi}) \cdot \left(C(\vec{x}(u, \vec{\pi})) + \lambda_i \cdot S_i w(s - u, \vec{x}(u, \vec{\pi})) \right) du,$$

in terms of the operators

$$(3.8) \quad S_i w(t, \vec{\pi}) \triangleq \int_{\mathbb{R}^d} w \left(t, \frac{\lambda_1 f_1(y) \pi_1}{\sum_{j \in E} \lambda_j f_j(y) \pi_j}, \dots, \frac{\lambda_n f_n(y) \pi_n}{\sum_{j \in E} \lambda_j f_j(y) \pi_j} \right) f_i(y) \nu(dy), \quad \text{for } i \in E.$$

The following lemmas provide basic properties of the operator J_0 .

Lemma 3.2. *If $w(\cdot, \cdot)$ is bounded, then so is $J_0 w(\cdot, \cdot)$ on $[0, T] \times D$. If $w_1(\cdot, \cdot) \leq w_2(\cdot, \cdot)$, then $J_0 w_1(\cdot, \cdot) \leq J_0 w_2(\cdot, \cdot)$. Moreover, if the mapping $\vec{\pi} \mapsto w(s, \vec{\pi})$ is convex for each $s \in [0, T]$, so is $\vec{\pi} \mapsto J_0 w(s, \vec{\pi})$ for each $s \in [0, T]$.*

Remark 3.1. *For a bounded continuous function $w(\cdot, \cdot)$ on $[0, T] \times D$, the mapping $t \rightarrow Jw(t, s, \vec{\pi})$ is continuous on $[0, s]$ and $\sup_{t \in [u, s]} Jw(t, s, \vec{\pi})$ is attained for all $u \in [0, s]$.*

Lemma 3.3. *The operator J_0 preserves the continuity. That is, if $w(\cdot, \cdot)$ is a continuous function defined on $[0, T] \times D$, then $J_0 w(\cdot, \cdot)$ is also continuous.*

Let us now define the sequence

$$(3.9) \quad v_0(s, \vec{\pi}) \triangleq H(\vec{\pi}), \quad \text{and} \quad v_{m+1}(s, \vec{\pi}) \triangleq J_0 v_m(s, \vec{\pi}), \quad \text{for } m \geq 0, \quad \text{on } [0, T] \times D.$$

Lemma 3.4. *The sequence $\{v_m(\cdot, \cdot)\}_{m \in \mathbb{N}}$ is non-decreasing, hence the pointwise limit $v(\cdot, \cdot) \triangleq \sup_{m \in \mathbb{N}} v_m(\cdot, \cdot)$ is well defined on $[0, T] \times D$. Each $v_m(\cdot, \cdot)$ is bounded and continuous on $[0, T] \times D$, and the mapping $\vec{\pi} \mapsto v_m(s, \vec{\pi})$ is convex for each $s \in [0, T]$.*

Proof. Note that $v_1(s, \vec{\pi}) = J_0 v_0(s, \vec{\pi}) = J_0 H(s, \vec{\pi}) = \sup_{t \in [0, s]} J_0 H(t, s, \vec{\pi}) \geq J_0 H(0, s, \vec{\pi}) = H(\vec{\pi})$. Let us assume that $v_m \geq v_{m-1}$ for some $m \in \mathbb{N}$. Then we get $v_{m+1}(s, \vec{\pi}) = J_0 v_m(s, \vec{\pi}) \geq J_0 v_{m-1}(s, \vec{\pi}) = v_m(s, \vec{\pi})$ where the inequality follows due to Lemma 3.2. Hence, the sequence is non-decreasing by induction.

The claim on continuity, boundedness and convexity clearly hold for $v_0(\cdot, \cdot) = H(\cdot)$. Then using Lemmas 3.2 and 3.3 it can be verified inductively that these properties also hold for each v_m . \square

Proposition 3.2. *The sequences defined in (3.2) and (3.9) coincide. That is, we have $v_m(\cdot, \cdot) = V_m(\cdot, \cdot)$ for every $m \in \mathbb{N}$.*

Corollary 3.1. *Propositions 3.1 and 3.2 imply $v(\cdot, \cdot) \triangleq \lim_{m \in \mathbb{N}} v_m(\cdot, \cdot) = \lim_{m \in \mathbb{N}} V_m(\cdot, \cdot) = V(\cdot, \cdot)$. By Lemma 3.4, each $V_m(\cdot, \cdot)$ is continuous on $[0, T] \times D$. Then, the uniform convergence in Proposition 3.1 implies that $V(\cdot, \cdot)$ is also continuous. Finally, as the upper envelope of convex mappings $\vec{\pi} \mapsto v_m(s, \vec{\pi}) = V_m(s, \vec{\pi})$, the mapping $\vec{\pi} \mapsto V(s, \vec{\pi})$ is again convex for each $s \in [0, T]$.*

Proposition 3.3 below characterizes the value function $V(\cdot, \cdot)$ as the fixed point of the operator J_0 defined in (3.5-3.7), which can also be thought of as the dynamic programming equation for the value function $V(\cdot, \cdot)$.

Proposition 3.3. *The value function satisfies $V(s, \vec{\pi}) = J_0 V(s, \vec{\pi})$, and it is the smallest bounded solution of this equation greater than $H(\cdot)$.*

Proof. Using Lemma 3.4 and Corollary 3.1 we get $V(s, \vec{\pi}) = v(s, \vec{\pi}) = \sup_{n \geq 1} v_n(s, \vec{\pi}) =$

$$\begin{aligned} \sup_{n \geq 1} \sup_{t \in [0, s]} Jv_{n-1}(t, s, \vec{\pi}) &= \sup_{t \in [0, s]} \sup_{n \geq 1} Jv_{n-1}(t, s, \vec{\pi}) = \sup_{t \in [0, s]} \sup_{n \geq 1} \mathbb{E}^{\vec{\pi}} \left[e^{-I(t)} \right] \cdot e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi})) \\ &\quad + \int_0^t e^{-\rho u} \sum_{i \in E} m_i(u, \vec{\pi}) \cdot \left(C(\vec{x}(u, \vec{\pi})) + \lambda_i \cdot S_i v_{n-1}(s - u, \vec{x}(u, \vec{\pi})) \right) du \\ &= \sup_{t \leq s} \mathbb{E}^{\vec{\pi}} \left[e^{-I(t)} \right] \cdot e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi})) \\ &\quad + \int_0^t e^{-\rho u} \sum_{i \in E} m_i(u, \vec{\pi}) \cdot \left(C(\vec{x}(u, \vec{\pi})) + \lambda_i \cdot S_i v(s - u, \vec{x}(u, \vec{\pi})) \right) du \\ &= \sup_{t \in [0, s]} Jv(t, s, \vec{\pi}) = \sup_{t \in [0, s]} JV(t, s, \vec{\pi}), \end{aligned}$$

where the fifth equality is from (3.7) and the sixth equality is by the bounded convergence theorem since we have $\|v_m(\cdot, \cdot)\| \leq \|v(\cdot, \cdot)\| \leq \|H(\cdot)\| + T\|C(\cdot)\|$ for all $m \in \mathbb{N}$.

Let $W(\cdot, \cdot)$ be another solution of $W(s, \vec{\pi}) = J_0 W(s, \vec{\pi})$, such that $W(s, \vec{\pi}) \geq H(\vec{\pi}) = v_0(s, \vec{\pi})$. Applying Remark 3.2 we obtain $W(s, \vec{\pi}) = J_0 W(s, \vec{\pi}) \geq \sup_{t \in [0, s]} Jv_0(t, s, \vec{\pi}) = v_1(s, \vec{\pi})$. By induction, $W(s, \vec{\pi}) \geq v_n(s, \vec{\pi})$ for all n and hence $W(s, \vec{\pi}) \geq \lim_{n \rightarrow \infty} v_n(s, \vec{\pi}) = V(s, \vec{\pi})$. \square

We finally close this section with the following result which will be useful in Section 4 in establishing an optimal stopping time.

Lemma 3.5. *For deterministic times $u \leq t \leq s$, and for a bounded function $w(\cdot, \cdot)$ we have*

$$(3.10) \quad Jw(t, s, \vec{\pi}) = Jw(u, s, \vec{\pi}) + \mathbb{P}^{\vec{\pi}} \{ \sigma_1 > u \} \cdot e^{-\rho u} \cdot \left(Jw(t - u, s - u, x(u, \vec{\pi})) - H(x(u, \vec{\pi})) \right)$$

Corollary 3.2. *Let w be a bounded function as in Lemma 3.5. Taking the supremum in (3.10) for fixed u and s we obtain*

$$\sup_{t \in [u, s]} Jw(t, s, \vec{\pi}) = Jw(u, s, \vec{\pi}) + \mathbb{P}^{\vec{\pi}} \{ \sigma_1 > u \} \cdot e^{-\rho u} \cdot \left(J_0 w(s - u, x(u, \vec{\pi})) - H(x(u, \vec{\pi})) \right),$$

where J_0 is as defined in (3.5).

4. AN OPTIMAL STRATEGY

Recall that the process $\vec{\Pi}$ has right-continuous paths (with left limits), and the functions $V(\cdot, \cdot)$ and $H(\cdot)$ are continuous due to Corollary 3.1. Hence the paths of the process $V(t, \vec{\Pi}_t) - H(\vec{\Pi}_t)$ are also right-continuous and have left limits. Therefore, for $\varepsilon \geq 0$ the random time

$$(4.1) \quad U_\varepsilon(s, \vec{\pi}) \triangleq \inf \left\{ t \in [0, s] : V(s - t, \vec{\Pi}_t) - \varepsilon \leq H(\vec{\Pi}_t) \right\}$$

is a well-defined \mathbb{F} -stopping time. Observe that we have $U_\varepsilon(s, \vec{\pi}) \wedge \sigma_1 = r_\varepsilon(s, \vec{\pi}) \wedge \sigma_1$, where

$$(4.2) \quad r_\varepsilon(s, \vec{\pi}) \triangleq \inf \{ t \in [0, s] : V(s - t, \vec{x}(t, \vec{\pi})) - \varepsilon \leq H(\vec{x}(t, \vec{\pi})) \},$$

which can be considered as the deterministic counterpart of (4.1).

Remark 4.1. For $r_\varepsilon(s, \vec{\pi})$ defined in (4.2) we have

$$(4.3) \quad \sup_{t \in [0, s]} JV(t, s, \vec{\pi}) = \sup_{t \in [r_\varepsilon(s, \vec{\pi}), s]} JV(t, s, \vec{\pi}).$$

Proof. For $t < r_\varepsilon(s, \vec{\pi})$, Proposition 3.3 and Corollary 3.2 give

$$JV(t, s, \vec{\pi}) = \sup_{u \in [t, s]} JV(u, s, \vec{\pi}) - \mathbb{P}^{\vec{\pi}}\{\sigma_1 > t\} e^{-\rho t} \left(V(s - t, \vec{x}(t, \vec{\pi})) - H(\vec{x}(t, \vec{\pi})) \right).$$

Since $t < r_\varepsilon(s, \vec{\pi})$ we have $V(s - t, \vec{x}(t, \vec{\pi})) - H(\vec{x}(t, \vec{\pi})) > \varepsilon$. Hence

$$\begin{aligned} JV(t, s, \vec{\pi}) &\leq \sup_{u \in [t, s]} JV(u, s, \vec{\pi}) - \varepsilon \mathbb{P}^{\vec{\pi}}\{\sigma_1 > t\} e^{-\rho t} \\ &\leq \sup_{u \in [0, s]} JV(u, s, \vec{\pi}) - \varepsilon \mathbb{P}^{\vec{\pi}}\{\sigma_1 > t\} e^{-\rho t} < \sup_{u \in [0, s]} JV(u, s, \vec{\pi}). \end{aligned}$$

Therefore the supremum in $\sup_{t \in [0, s]} JV(t, s, \vec{\pi})$ must be achieved on $[r_\varepsilon(s, \vec{\pi}), s]$ and (4.3) follows. \square

Proposition 4.1. The stopping time $U_\varepsilon(s, \vec{\pi})$ defined in (4.1) is an ε -stopping time for the problem in (2.3), i.e.,

$$(4.4) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} H(\vec{\Pi}(U_\varepsilon(s, \vec{\pi}))) \right] \geq V(s, \vec{\pi}) - \varepsilon,$$

for all $\varepsilon \geq 0$ and $(s, \vec{\pi}) \in [0, T] \times D$.

Before proceeding with the proof of Proposition 4.1, we first state an immediate consequence of this result.

Corollary 4.1. The stopping time $U_0(T, \vec{\pi})$ is an optimal rule for the stopping problem of (2.3), and the pair $(U_0(T, \vec{\pi}), d(U_0(T, \vec{\pi})))$ is an optimal admissible strategy for the problem in (1.5).

Proof of Proposition 4.1. Let us define

$$(4.5) \quad Z_t \triangleq \int_0^t e^{-\rho u} C(\vec{\Pi}_u) du + e^{-\rho t} V(s - t, \vec{\Pi}_t), \quad t \in [0, s],$$

which is a bounded process on $t \in [0, s] \subseteq [0, T]$. We will show that the stopped process $\{Z_{t \wedge U_\varepsilon(s, \vec{\pi})}\}_{t \in [0, s]}$ is a martingale and satisfies

$$(4.6) \quad \mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi})}] = Z_0 = V(s, \vec{\pi}).$$

The process Z captures the natural idea that one should not stop as long as the value function (i.e. the continuation value) is larger than the immediate reward. Note that ε -optimality of $U_\varepsilon(s, \vec{\pi})$ follows easily from (4.6) since this equality would imply $V(s, \vec{\pi}) = \mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi})}] =$

$$(4.7) \quad \begin{aligned} \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} V(s - U_\varepsilon(s, \vec{\pi}), \vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) \right] &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + \right. \\ &\quad \left. e^{-\rho U_\varepsilon(s, \vec{\pi})} \left(H(\vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) + \varepsilon \right) \right] \leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} H(\vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) \right] + \varepsilon, \end{aligned}$$

due to regularity of the paths $t \mapsto V(t, \vec{\Pi}_t) - H(\vec{\Pi}_t)$. In the remainder of the proof we will show (4.6) by establishing

$$(4.8) \quad \mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m}] = Z_0, \quad \text{for } m = 1, 2, \dots,$$

inductively. After taking the limit as $m \rightarrow \infty$ in the equality above, we will then obtain (4.6) due to bounded convergence theorem.

First, consider the equality (4.8) for $m = 1$. Recall that $U_\varepsilon(s, \vec{\pi}) \wedge \sigma_1 = r_\varepsilon(s, \vec{\pi}) \wedge \sigma_1$. Then $\mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_1}] = \mathbb{E}^{\vec{\pi}}[Z_{r_\varepsilon(s, \vec{\pi}) \wedge \sigma_1}] =$

$$\begin{aligned} & \mathbb{E}^{\vec{\pi}} \left[\int_0^{r_\varepsilon(s, \vec{\pi}) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\sigma_1 \leq r_\varepsilon(s, \vec{\pi})\}} \cdot e^{-\rho \sigma_1} V(s - \sigma_1, \vec{\Pi}_{\sigma_1}) + 1_{\{\sigma_1 > r_\varepsilon(s, \vec{\pi})\}} \cdot e^{-\rho r_\varepsilon(s, \vec{\pi})} H(\vec{\Pi}_{r_\varepsilon(s, \vec{\pi})}) \right. \\ & \quad \left. + 1_{\{\sigma_1 > r_\varepsilon(s, \vec{\pi})\}} \cdot e^{-\rho r_\varepsilon(s, \vec{\pi})} \left(V(s - r_\varepsilon(s, \vec{\pi}), \vec{\Pi}_{r_\varepsilon(s, \vec{\pi})}) - H(\vec{\Pi}_{r_\varepsilon(s, \vec{\pi})}) \right) \right] \\ &= JV(r_\varepsilon(s, \vec{\pi}), s, \vec{\pi}) + e^{-\rho r_\varepsilon(s, \vec{\pi})} \cdot \mathbb{P}^{\vec{\pi}}\{\sigma_1 > r_\varepsilon(s, \vec{\pi})\} \cdot (V(s - r_\varepsilon(s, \vec{\pi}), \vec{x}(r_\varepsilon(s, \vec{\pi}), \vec{\pi})) - H(\vec{x}(r_\varepsilon(s, \vec{\pi}), \vec{\pi}))) \\ &= \sup_{u \in [r_\varepsilon(s, \vec{\pi}), s]} JV(u, s, \vec{\pi}), \end{aligned}$$

where we used Proposition 3.3 and Corollary 3.2 for the last equality. By Remark 4.1, we get

$$\mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_1}] = \sup_{u \in [r_\varepsilon(s, \vec{\pi}), s]} JV(u, s, \vec{\pi}) = J_0 V(s, \vec{\pi}) = V(s, \vec{\pi}) = Z_0,$$

and this establishes the result for $m = 1$.

Now suppose by induction that (4.8) is true for $m \geq 1$ and consider the equality

$$\begin{aligned} (4.9) \quad \mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}}] &= \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) < \sigma_1\}} Z_{U_\varepsilon(s, \vec{\pi})} + 1_{\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}} \right] \\ &= \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) < \sigma_1\}} \left(\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} V(s - U_\varepsilon(s, \vec{\pi}), \vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) \right) \right. \\ & \quad \left. + 1_{\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} \left(\int_0^{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}} V(s - U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}, \vec{\Pi}_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}}) \right) \right]. \end{aligned}$$

On the event $\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}$, we have $U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1} = \sigma_1 + [U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m] \circ \theta_{\sigma_1}$, where θ is the time-shift operator on Ω ; i.e., $X_t \circ \theta_s = X_{t+s}$. Using the strong Markov property of $\vec{\Pi}$, equation (4.9) becomes $\mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}}] =$

$$\begin{aligned} (4.10) \quad \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) < \sigma_1\}} \left(\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} V(s - U_\varepsilon(s, \vec{\pi}), \vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) \right) \right. \\ \left. + \int_0^{\sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} e^{-\rho \sigma_1} f(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right], \end{aligned}$$

where $f(u, \vec{\pi}) \triangleq$

$$(4.11) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m} V(u - U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m, \vec{\Pi}_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_m}) \right] = V(u, \vec{\pi}),$$

by the induction hypothesis for m . Combining (4.10) and (4.11) we get $\mathbb{E}^{\vec{\pi}}[Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}}] =$

$$\begin{aligned} & \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) < \sigma_1\}} \left(\int_0^{U_\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_\varepsilon(s, \vec{\pi})} V(s - U_\varepsilon, \vec{\Pi}_{U_\varepsilon(s, \vec{\pi})}) \right) \right] \\ & \quad + \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} \left(\int_0^{\sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \sigma_1} V(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right) \right] \\ & = \mathbb{E}^{\vec{\pi}} \left[1_{\{U_\varepsilon(s, \vec{\pi}) < \sigma_1\}} Z_{U_\varepsilon(s, \vec{\pi})} + 1_{\{U_\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} Z_{\sigma_1} \right] = \mathbb{E}^{\vec{\pi}} [Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_1}] = Z_0, \end{aligned}$$

where the last equality follows from our result for $m = 1$. Hence we have $\mathbb{E}^{\vec{\pi}} [Z_{U_\varepsilon(s, \vec{\pi}) \wedge \sigma_{m+1}}] = Z_0$ and this gives (4.8) for $m + 1$. \square

4.1. Stopping and continuation regions. Let

$$(4.12) \quad \begin{aligned} \mathcal{C}_T &\triangleq \{(s, \vec{\pi}) \in [0, T] \times D : V(s, \vec{\pi}) > H(\vec{\pi})\}, \\ \Gamma_T &\triangleq \{(s, \vec{\pi}) \in [0, T] \times D : V(s, \vec{\pi}) = H(\vec{\pi})\} \end{aligned}$$

denote the continuation and stopping regions respectively. The stopping region can further be decomposed as the union $\cup_{k \in \mathcal{A}} \Gamma_{T,k}$ of the regions

$$(4.13) \quad \Gamma_{T,k} \triangleq \{(s, \vec{\pi}) \in [0, T] \times D : V(s, \vec{\pi}) = H_k(\vec{\pi})\}, \quad k \in \mathcal{A},$$

where H_k is defined in (2.4). Corollary 4.1 states that in the optimal solution $(U_0(T, \vec{\pi}), d(U_0(T, \vec{\pi})))$, one observes the process $\vec{\Pi}$ until $U_0(T, \vec{\pi})$, whence it enters the region Γ_T . At this time, if $\vec{\Pi}$ is in the set $\Gamma_{T,k}$ we take $d(U_0(T, \vec{\pi})) = k$; that is, we select the k 'th action in the action set \mathcal{A} .

Remark 4.2. The definition of the value function V in (2.3) implies that the mapping $s \mapsto V(s, \vec{\pi})$ is non-decreasing. Therefore if $(s, \vec{\pi}) \in \Gamma_{T,k}$ for some $(s, \vec{\pi}) \in [0, T] \times D$, then we have $(t, \vec{\pi}) \in \Gamma_{T,k}$ for all $t \leq s$. In other words, each region $\Gamma_{T,k}$ is growing and the continuation region \mathcal{C}_T is shrinking as time to maturity decreases.

Remark 4.3. For fixed $s \leq T$, let $(s, \vec{\pi}_1)$ and $(s, \vec{\pi}_2)$ be two points in the region $\Gamma_{T,k}$, and let $\alpha \in (0, 1)$. As the upper envelope of convex mappings $\vec{\pi} \rightarrow v_m(s, \vec{\pi})$ (see Lemma 3.4 and Corollary 3.1), the mapping $\vec{\pi} \rightarrow V(s, \vec{\pi})$ is convex for each $s \in [0, T]$. Using this property we obtain

$$\begin{aligned} H_k(\alpha \cdot \vec{\pi}_1 + (1 - \alpha) \cdot \vec{\pi}_2) &\leq V(s, \alpha \cdot \vec{\pi}_1 + (1 - \alpha) \cdot \vec{\pi}_2) \leq \alpha \cdot V(s, \vec{\pi}_1) + (1 - \alpha) \cdot V(s, \vec{\pi}_2) \\ &= \alpha \cdot H_k(\vec{\pi}_1) + (1 - \alpha) \cdot H_k(\vec{\pi}_2) = H_k(\alpha \cdot \vec{\pi}_1 + (1 - \alpha) \cdot \vec{\pi}_2), \end{aligned}$$

which implies that $(s, \alpha \cdot \vec{\pi}_1 + (1 - \alpha) \cdot \vec{\pi}_2) \in \Gamma_{T,k}$, and the region $\Gamma_{T,k} \cap (\{s\} \times D)$ is convex for each fixed $s \leq T$ and $k \in \mathcal{A}$.

Remark 4.4. The stopping region is never empty since the decision maker has to select an action eventually, the latest at the terminal time T . That is, $\Gamma_T \supseteq \{(0, \vec{\pi}); \vec{\pi} \in D\} \neq \emptyset$. The region $\{(s, \vec{\pi}) \in \Gamma_T : s > 0\}$ may however be empty. In an example where $\min_{i \in E} c_i > 0$ and $\mu_{k,i}$'s are all the same it is never optimal to stop prior to terminal time T .

Note that the region $\{(s, \vec{\pi}) \in \Gamma_T : s > 0\}$ may be non-empty but still may have an empty interior. For example, let us consider the hypothesis testing in (1.3). In this *minimization* problem, all the states of the unobservable Markov process are absorbing, and each component $\Pi_t^{(i)} = \mathbb{P}\{M_t = i | \mathcal{F}_t^X\} = \mathbb{P}\{M_0 = i | \mathcal{F}_t^X\}$ of process $\vec{\Pi}$ is a martingale. Since the terminal reward function of the corresponding stopping problem (see (2.4)) $H(\cdot) = \min_{k \in E} H_k(\cdot)$ is concave, the process $H(\vec{\Pi}_t)$ is a supermartingale on $[0, T]$. If we select $\rho = 0$ and $c_i = 0$ for all $i \in E$ in (1.3), it is therefore never optimal to stop early on the interior of $\{(s, \vec{\pi}) \in \Gamma_T : s > 0\}$. In this case, there is no penalty associated with a delay in the decision. Hence the DM will choose to observe it as much as possible prior to a decision unless she knows for sure which hypothesis is correct.

Lemma 4.1. *For $i \in E$, let $\mathcal{A}^*(i) \triangleq \{k \in \mathcal{A} : \mu_{k,i} = \max_{j \in \mathcal{A}} \mu_{j,i}\}$. If the inequality $c_i - \rho\mu_{k,i} + \sum_{j \neq i} (\mu_{k,j} - \mu_{k,i})q_{i,j} > 0$ holds for all $k \in \mathcal{A}^*(i)$, then there exists $\pi_i^c < 1$ such that $\{(s, \vec{\pi}) \in (0, T] \times D : \pi_i \geq \pi_i^c\} \subseteq \mathcal{C}_T$. Moreover, π_i^c can be selected independent of T .*

If the hidden process M is known to be in state $i \in E$, then the expression $-\rho\mu_{k,i}$ is the instantaneous decay of the payoff from selecting action $k \in \mathcal{A}$ immediately, and c_i is the instantaneous cost of waiting. Moreover, under action $k \in \mathcal{A}$, the term $\sum_{j \neq i} (\mu_{k,j} - \mu_{k,i})q_{i,j}$ is the marginal rate of return from waiting for the hidden process M to jump to another state. Therefore the sum of these three terms appearing in Lemma 4.1 is the instantaneous net return enjoyed by the DM under action $k \in \mathcal{A}$. Lemma 4.1 indicates that if there is strong posteriori evidence that M is in state i , and if the instantaneous net return is positive under all favorable actions (whose terminal reward H_k dominates others around the i 'th corner of D), the decision maker should not stop at that point (unless $T = 0$).

4.2. Stopping regions for reward maximization with running cost. Here, we consider the problem in (2.3) with the assumption $c_i \leq 0$ (running costs) for $i \in E$, and $\bar{\mu} \triangleq \max_{k,i} \mu_{k,i} > 0$ (terminal rewards). The second condition is not restrictive if $\rho = 0$ since we can always add (and subtract) the same constant to (and from) the terminal reward function.

Let us define

$$(4.14) \quad I^* \triangleq \{i \in E : \max_{k \in \mathcal{A}} \mu_{k,i} = \bar{\mu}\},$$

which is the set of the states of M , at which the DM can get the highest terminal reward. Since $c_i \leq 0$ for all $i \in E$, we have $\cup_{i \in I^*} \{(s, \vec{\pi}) : s \in [0, T], \pi_i = 1\} \subset \Gamma_T$. That is the DM stops whenever the process $\vec{\Pi}$ reaches a point of global maximum of the terminal reward function $H(\cdot)$.

In general, if there is a penalty associated with waiting, we expect that it is optimal to stop on the points $(s, \vec{\pi})$ for which the “best” component π_i , $i \in I^*$, is sufficiently high, for any $s > 0$. Lemma 4.2 provides a sufficient condition for this to be true. It implies that if the discount rate is strictly positive, or if the cost of waiting for the highest reward is strictly positive, then we stop whenever π_i , for $i \in I^*$, is relatively high regardless of the remaining time to maturity.

Lemma 4.2. *Let $i \in I^*$. If $\rho > 0$, or $c_i < 0$, then there exists a number $\pi_i^s < 1$ such that*

$$\Gamma_T \supseteq \{(s, \vec{\pi}) \in [0, T] \times D : \pi_i \geq \pi_i^s\},$$

and the value of π_i^s can be selected free of the time to maturity T .

Remark 4.5. If $H(\cdot) \geq 0$, the statement of the stopping problem in (2.3) implies that the value function V is non-increasing as a function of the discount factor ρ . If we denote the dependence of the stopping region on ρ with $\Gamma_T(\rho)$, then we have $\Gamma_T(\rho_1) \subseteq \Gamma_T(\rho_2)$ whenever $\rho_1 \leq \rho_2$. Moreover, the dynamics of the process $\vec{\Pi}$ are independent of ρ and $U_0(s, \vec{\pi})$ is the hitting time of $\vec{\Pi}$ to Γ_T . Therefore, the time that the DM can afford for observing the process X in the presence of a lower discount factor is no less than that spent under heavier discounting.

A similar claim also holds for dependence of $U_0(s, \vec{\pi})$ and Γ_T on the running costs c_i . Namely, an observer with lower (in absolute value) running costs stops no sooner than another one with heavier running costs.

4.3. A nearly-optimal strategy. On a practical level, one cannot compute V directly, but instead computes the approximate value functions V_m 's defined in (3.2) and employs the corresponding nearly-optimal strategies (see 4.15). It is therefore important to know the error associated with this approximation.

For a given error level $\varepsilon > 0$, let us fix

$$m = \inf \left\{ k \in \mathbb{N} : (T\|C\| + 2\|H\|) \left(\frac{\bar{\lambda}T}{k-1} \right)^{1/2} \cdot \left(\frac{\bar{\lambda}}{2\rho + \bar{\lambda}} \right)^{k/2} \leq \varepsilon/2 \right\},$$

such that $\|V_m - V\| \leq \varepsilon/2$ on $[0, T] \times D$ via (3.3). Next, let us define the stopping times

$$(4.15) \quad U_{\varepsilon/2}^{(m)}(s, \vec{\pi}) \triangleq \inf \{ t \in [0, s] : V_m(s, \vec{\Pi}_t) - \varepsilon/2 \leq H(\vec{\Pi}_t) \}.$$

The regularity of the paths $t \mapsto \vec{\Pi}_t$ implies that $V \left(U_{\varepsilon/2}^{(m)}(s, \vec{\pi}), \vec{\Pi}_{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} \right) - H \left(\vec{\Pi}_{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} \right) \leq \varepsilon$.

Then the arguments in the proof of Proposition 4.1 (see (4.5), (4.6), and (4.7)) can easily be modified to show that

$$(4.16) \quad \begin{aligned} V(s, \vec{\pi}) &= \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} V \left(s - U_{\varepsilon/2}^{(m)}(s, \vec{\pi}), \vec{\Pi}_{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} \right) \right] \\ &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} H \left(\vec{\Pi}_{U_{\varepsilon/2}^{(m)}(s, \vec{\pi})} \right) \right] + \varepsilon. \end{aligned}$$

Hence, if we apply the admissible strategy $(U_{\varepsilon/2}^{(m)}(T, \vec{\pi}), d(U_{\varepsilon/2}^{(m)}(T, \vec{\pi})))$, which requires computing (3.2) only up to m defined above, the resulting error is no more than ε .

4.4. Infinite horizon problem as an approximation. In general, if there is a strict penalty for waiting, it is likely that the DM will make a decision prior to the final time T for moderate or large values of T . In this case, the constraint $\tau \leq T$ in (2.3) is of less importance, and one essentially faces an *infinite horizon* stopping problem. Solving the infinite horizon problem can be computationally more appealing since we eliminate the time-dimension of the state space $[0, T] \times D$. Below, we show that the value function of the finite-horizon problem converges uniformly to that of the infinite horizon under the assumption

$$(4.17) \quad \text{“either } \rho > 0 \text{” or “} \max_{i \in E} c_i < 0 \text{”}.$$

The infinite horizon problem is defined as in (2.3) (and (1.5)) by removing the constraint $\tau \leq T$. With the notation in (2.3), let $V(\infty, \vec{\pi})$ be the value function of this stopping problem.

Lemma 4.3. *As $T \nearrow \infty$, the function $V(T, \vec{\pi})$ converges to $V(\infty, \vec{\pi})$ uniformly on D , and we have*

$$(4.18) \quad V(T, \vec{\pi}) \leq V(\infty, \vec{\pi}) \leq V(T, \vec{\pi}) + \text{Err}(T), \quad \text{for all } \vec{\pi} \in D \text{ and } T \geq 0,$$

where

$$\text{Err}(T) \triangleq \begin{cases} e^{-\rho T} (\|C\| + 2 \cdot \|H\|) & , \quad \text{if } \rho > 0 \\ \frac{2 \cdot \|H\|}{T} \frac{(\min_{k,i} \mu_{k,i} - \max_{k,i} \mu_{k,i})}{\max_{i \in E} c_i} & , \quad \text{if } \rho = 0 \text{ and } \max_{i \in E} c_i < 0. \end{cases}$$

The explicit error bounds for the rate of convergence allows to approximate $V(T, \cdot)$ with the value function of the infinite horizon problem when T is large. The function $V(\infty, \vec{\pi})$ can be computed sequentially as in Section 3. That is, if we define the non-decreasing sequence

$$(4.19) \quad V_m(\infty, \vec{\pi}) \triangleq \sup_{\tau \geq 0} \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right], \quad m \in \mathbb{N},$$

then it can be shown that the elements of this sequence can be computed by applying a functional operator \hat{J}_0 , which is obtained from the operator J_0 in (3.7) after replacing the constraint $t \in [0, s]$ with $t \geq 0$. Also, note that the new operator \hat{J}_0 is defined on the domain of functions defined on

D only. The proof of these statements can be obtained by modifying the arguments of Section 3, or those in [12, Section 3]. Moreover, following the proof of Proposition 3.1 and the arguments of Section 4.3, we have

$$\|V_m(\infty, \cdot) - V(\infty, \vec{\pi})\| \leq Err_\infty(m) \triangleq \begin{cases} \left(\frac{\bar{\lambda}}{\rho + \bar{\lambda}}\right)^m, & \text{if } \rho > 0, \\ \left(\frac{\max_{k,i} \mu_{k,i}}{\max_{i \in E} c_i} \cdot \frac{\bar{\lambda}}{m-1}\right)^{1/2}, & \text{if } \rho = 0 \text{ and } \max_{i \in E} c_i < 0, \end{cases}$$

and the stopping time

$$(4.20) \quad U_\varepsilon^{(m)}(\infty, \vec{\pi}) \triangleq \inf \left\{ t \geq 0 : V_m(\infty, \vec{\Pi}_t) - \varepsilon \leq H(\vec{\Pi}_t) \right\}$$

is ε -optimal for the infinite horizon problem (see also [12, Section 4.1]).

Note that for large m , the function $V_m(\infty, \cdot)$ approximates the function $V(\infty, \cdot)$, and for large T , $V(\infty, \cdot)$ is a good approximation for $V(T, \cdot)$. However, the stopping rule in (4.20) is not a good substitute for the optimal time $U_0(T, \vec{\pi})$ since the former may not be less than T almost surely. Moreover, since $U_\varepsilon^{(m)}(\infty, \vec{\pi})$ may be greater than $U_0(T, \vec{\pi})$, Proposition 4.1 is not necessarily true. In particular, the martingale property (4.6) may fail. Nevertheless, if we apply the rule $U_\varepsilon^{(m)}(\infty, \vec{\pi}) \wedge T$, we can still control the error for large T . Indeed, in Appendix A2, we show that

$$(4.21) \quad V(T, \vec{\pi}) \leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon^{(m)}(\infty, \vec{\pi}) \wedge T} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho(U_\varepsilon^{(m)}(\infty, \vec{\pi}) \wedge T)} H(\vec{\Pi}_{U_\varepsilon^{(m)}(\infty, \vec{\pi}) \wedge T}) \right] \\ + \varepsilon + Err_\infty(m) + Err_\infty(0) \cdot Err(T).$$

Hence, if T is large enough (so that $Err_\infty(0) \cdot Err(T)$ is small), by taking ε in (4.20) small for a large value of m , the error associated with applying $U_\varepsilon^{(m)} \wedge T$ can be reduced to acceptable levels.

5. DISCRETE INFORMATION COSTS

As the case studies of Section 1 demonstrate, the objective function in (1.5) is applicable to a variety of economic settings. This has allowed us to provide a unified treatment of many disparate models. Returning to the economic interpretation of the running costs appearing in the first term in (1.5), in a typical setting they represent *information acquisition expenses*, such as observation expenses, subscription costs to market data and holding outlays. In such a case, it is natural to model the total cost incurred by decision time τ as the sum $\int_0^\tau e^{-\rho t} c dt$ where c is interpreted as nominal running cost and ρ is the interest rate.

Alternatively, the costs can correspond to *opportunity costs*, e.g. if M is the profitability of a new product then the opportunity costs of not launching the product should depend on $\{M_t\}_{t \in [0, \tau]}$. This motivates the consideration of $\int_0^\tau e^{-\rho t} c_i 1_{\{M_t = i\}} dt$ where $c_i \in \mathbb{R}$ and ρ can again be interpreted as the discount factor.

Finally, observation costs may be discrete and be incurred only when new information arrives. This, for example, happens if new information corresponds to opportunities lost (e.g. deals signed by competitors), leading to a cost structure of the form $\sum_{j=1}^{N_\tau} e^{-\rho \sigma_j} K(Y_j)$. Here, N_τ is the number of arrivals by time τ , (σ_j, Y_j) are the arrival times and marks respectively, and $K(Y_j)$ is the cost incurred upon an arrival of size Y_j (with $K : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying $\nu_i K^+ \triangleq \int_{\mathbb{R}^d} K^+(y) \nu_i(dy) < \infty$, $\forall i \in E$).

In the third case, one deals with the objective function

$$(5.1) \quad \hat{U}(T, \vec{\pi}) \triangleq \sup_{\tau \leq T, d \in \mathcal{F}_\tau^X} \mathbb{E}^{\vec{\pi}} \left[\sum_{j=1}^{N_\tau} e^{-\rho \sigma_j} K(Y_j) + e^{-\rho \tau} \sum_{k=1}^a 1_{\{d=k\}} \left(\sum_{i \in E} \mu_{k,i} \cdot 1_{\{M_\tau=i\}} \right) \right],$$

by solving the equivalent stopping problem

$$\hat{V}(T, \vec{\pi}) \triangleq \sup_{\tau \leq T} \mathbb{E}^{\vec{\pi}} \left[\sum_{j=1}^{N_\tau} e^{-\rho \sigma_j} K(Y_j) + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right],$$

as in Proposition 2.3. One can verify that the sequential approximation method of Section 3 holds for the function \hat{V} . Namely, if we define the sequence

$$\hat{V}_m(s, \vec{\pi}) \triangleq \sup_{\tau \leq s} \mathbb{E}^{\vec{\pi}} \left[\sum_{j=1}^{m \wedge N_\tau} e^{-\rho \sigma_j} K(Y_j) + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right], \quad m \in \mathbb{N},$$

it can be shown (see (3.5-3.8), Proposition 3.2) that we have $\hat{V}_{m+1}(s, \vec{\pi}) = \hat{J}_0 \hat{V}_m(s, \vec{\pi})$ where the operator \hat{J}_0 is defined as

$$\begin{aligned} \hat{J}_0 w(s, \vec{\pi}) &= \sup_{t \in [0, s]} \mathbb{E}^{\vec{\pi}} \left[e^{-I(t)} \cdot e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi})) \right. \\ &\quad \left. + \int_0^t e^{-\rho u} \sum_{i \in E} m_i(t, \vec{\pi}) \cdot \lambda_i \left(\int_{\mathbb{R}^d} K(y) \nu_i(dy) + S_i w(s-u, \vec{x}(u, \vec{\pi})) \right) du, \right] \end{aligned}$$

for a bounded function $w : [0, T] \times D \mapsto \mathbb{R}$.

Clearly $\{V_m\}_{m \geq 0}$ is an increasing sequence. Using the inequality $\mathbb{E} \left[\sum_{j=1}^{N_T} K^+(Y_j) \right] \leq (\max_{i \in E} \lambda_i) T \cdot (\max_{i \in E} \nu_i K^+)$ and the truncation arguments in the proof of Proposition 3.1, one can show that the sequence converges to \hat{V} uniformly with the error bound

$$0 \leq V - V_m \leq \left((\max_{i \in E} \lambda_i) T \cdot (\max_{i \in E} \nu_i K^+) + 2\|H\| \right) \left(\frac{\bar{\lambda} T}{m-1} \right)^{1/2} \left(\frac{\bar{\lambda}}{2\rho + \bar{\lambda}} \right)^{m/2}.$$

Arguments in Sections 3 and 4 can then be replicated to conclude that

$$\mathbb{E}^{\vec{\pi}} \left[\sum_{j=1}^{N_{\hat{U}_\varepsilon(s, \vec{\pi})}} e^{-\rho \sigma_j} K(Y_j) + e^{-\rho \hat{U}_\varepsilon(s, \vec{\pi})} H(\vec{\Pi}(\hat{U}_\varepsilon(s, \vec{\pi}))) \right] \geq \hat{V}(s, \vec{\pi}) - \varepsilon,$$

for the stopping time $\hat{U}_\varepsilon(s, \vec{\pi}) \triangleq \inf \left\{ t \in [0, s] : \hat{V}(s-t, \vec{\Pi}_t) - \varepsilon \leq H(\vec{\Pi}_t) \right\}$. Hence, the admissible strategy $(\hat{U}_\varepsilon(s, \vec{\pi}), d(\hat{U}_\varepsilon(s, \vec{\pi})))$ is an optimal strategy for the problem in (5.1), as expected.

Furthermore, other results of Section 4 can be adjusted for this new objective function. Below, we summarize these results in a remark.

Remark 5.1. Let $\nu_j K \triangleq \int_{\mathbb{R}^d} K(y) \nu_j(dy)$, for $j \in E$.

- (i) For a given index $i \in E$, Define $\mathcal{A}^*(i) \triangleq \{k \in \mathcal{A} : \mu_{k,i} = \max_{j \in \mathcal{A}} \mu_{j,i}\}$ as in Lemma 4.1. If $-\rho \mu_{k,i} + \lambda_i \cdot \nu_i K + \sum_{j \neq i} (\mu_{k,j} - \mu_{k,i}) q_{i,j} > 0$ holds for all $k \in \mathcal{A}^*(i)$, then there exists some $\hat{\pi}_i^c < 1$ (for all $T > 0$) such that it is optimal to continue on the region $\{(0, T] \times D; \pi_i \geq \hat{\pi}_i^c\}$.
- (ii) Assume $\nu_j K \leq 0$ for all $j \in E$, and $\bar{\mu} \triangleq \max_{k,i} \mu_{k,i} > 0$, and let I^* be as in (4.14). For $i \in I^*$, if $\nu_i K < 0$ or $\rho > 0$ there exists a number $\hat{\pi}_i^s < 1$ (free of T) such that it is optimal to stop at the points $\vec{\pi}$ for which $\pi_i \geq \hat{\pi}_i^s$. That is: $\Gamma_{T,i} \supseteq \{[0, T] \times D; \pi_i \geq \hat{\pi}_i^s\}$ for all $T \geq 0$.

- (iii) In the case where $\nu_j K \leq 0$ for all $j \in E$, and $H(\cdot) \geq 0$, the stopping region is monotone in ρ and $\nu_j K$, for $j \in E$. Namely, if we increase one of these factors in absolute terms (keeping everything else fixed), the stopping region expands, and the DM is forced to make a decision sooner.
- (iv) For a given $\varepsilon > 0$, let $m \in \mathbb{N}$ such that $\|\hat{V}(T, \cdot) - \hat{V}(T, \cdot)\| \leq \varepsilon/2$. Then the stopping time $\hat{U}_{\varepsilon/2}^{(m)}(s, \vec{\pi}) \triangleq \inf \left\{ t \in [0, T] : \hat{V}_m(T - t, \vec{\Pi}_t) - \varepsilon \leq H(\vec{\Pi}_t) \right\}$ gives an ε -optimal strategy.
- (v) If “ $\rho > 0$ ” or “ $K(\cdot) \leq 0$ with $\max_{i \in E} \nu_i K(\cdot) < 0$ ”, then $\hat{V}(T, \cdot) \nearrow \hat{V}(\infty, \cdot)$ uniformly as in (4.18) if we redefine

$$Err(T) \triangleq \left\{ \begin{array}{ll} e^{-\rho T} \left(\max_{i \in E} \lambda_i \cdot \max_{i \in E} \nu_i K^+ + 2 \cdot \|H(\cdot)\| \right) & , \quad \text{if } \rho > 0 \\ \frac{2 \cdot \|H(\cdot)\|}{T} \frac{(\min_{k,i} \mu_{k,i} - \max_{k,i} \mu_{k,i})}{\min_{i \in E} \lambda_i \cdot \max_{i \in E} \nu_i K} & , \quad \text{if } \rho = 0, K(\cdot) \leq 0 \text{ and } \max_{i \in E} \nu_i K < 0. \end{array} \right\}$$

6. EXAMPLES

Below we provide numerical examples illustrating the use of our sequential approximation approach developed in Section 3. In each example, we approximate the value function by repeatedly (finitely many times) applying the operator J in (3.5) starting with the initial function $H(\cdot)$. We set the number of iterations $m \in \mathbb{N}$ such that the error $\|V_m(\cdot) - V(\cdot)\|$ is negligible (see (3.3)).

6.1. Insurance launch. Our first example illustrates profit maximization with information cost, which is the first example in Section 1.1. Here, M_t represents the state of the economy with three major states $E = \{1, 2, 3\} \equiv \{Boom, Growth, Recession\}$, and with the generator

$$Q = \begin{pmatrix} -4 & 3 & 1 \\ 2 & -4 & 2 \\ 0 & 3 & -3 \end{pmatrix}.$$

Let $\vec{\lambda} = [\lambda_1, \lambda_2, \lambda_3] = [1, 2, 5]$ and $\vec{\nu} = [\nu_1, \nu_2, \nu_3] = [Gamma(3, 2), Gamma(4, 2), Gamma(5, 2)]$. Conditional on the state of M being $i \in E$, the frequency of claims is λ_i and their common distribution is ν_i . Here, we consider the objective function in (1.1) with $\vec{\mu} \equiv [\mu_B, \mu_G, \mu_R] = [6, 1, -3]$, $\rho = 0.1$ and $c = -0.3$. As before, $d = 1$ represents the decision to launch the new policy; $d = 0$ represents the decision to abandon, and does not involve any cashflows. The horizon is taken to be $T = 0.8$ (whose unit is to be consistent with that of λ_i 's; e.g., if λ_i is in “customers per month”, T is in months).

For this example, we discretized $D = \{\vec{\pi} \in \mathbb{R}_+^3 : \pi_B + \pi_G + \pi_R = 1\}$ using 100 grid points in each dimension and computed V_m such that $\|V_m - V_{m-1}\| \leq 10^{-4}$. The triangular regions in Figure 2 show the region D . The corners $\{B, G, R\}$ corresponds to points where the states $\{Boom, Growth, Recession\}$ have posterior probabilities equal to 1 respectively. The left panel of Figure 2 shows the value function $V(0.8, \vec{\pi})$ and the shaded region is $\{\vec{\pi} \in D : V(0.8, \vec{\pi}) = H(\vec{\pi})\}$. Recall that it is optimal to stop as soon as $V(T - t, \vec{\Pi}_t) = H(\vec{\Pi}_t)$ and the corresponding stopping region is time-dependent. The right panel of Figure 2 illustrates this point by varying the problem horizon T . As expected from Remark 4.2, when T decreases, stopping regions expand. In particular, we see that with very little time left ($T = 0.1$ and $T = 0.2$), it is optimal to stop whenever π_B (where action $d = 1$ is chosen) or π_R is high (where quitting $d = 0$ is optimal). For longer horizons, the DM can afford to wait for favorable circumstances and release the product then. That is, stopping and selecting $d = 0$ is never optimal when time-to-maturity is not small. Also note that the terminal reward associated with $d = 1$ is higher than that of $d = 0$ around the corner G . Moreover, with the notation in Lemma 4.1 we have $r_G = c_G - \rho \mu_G + (\mu_B - \mu_G) q_{G,B} + (\mu_R - \mu_G) q_{G,R} = 1.6 > 0$. Then by Lemma 4.1, it is never optimal to stop around the corner G (unless $T = 0$) as shown the in right panel of Figure 2.

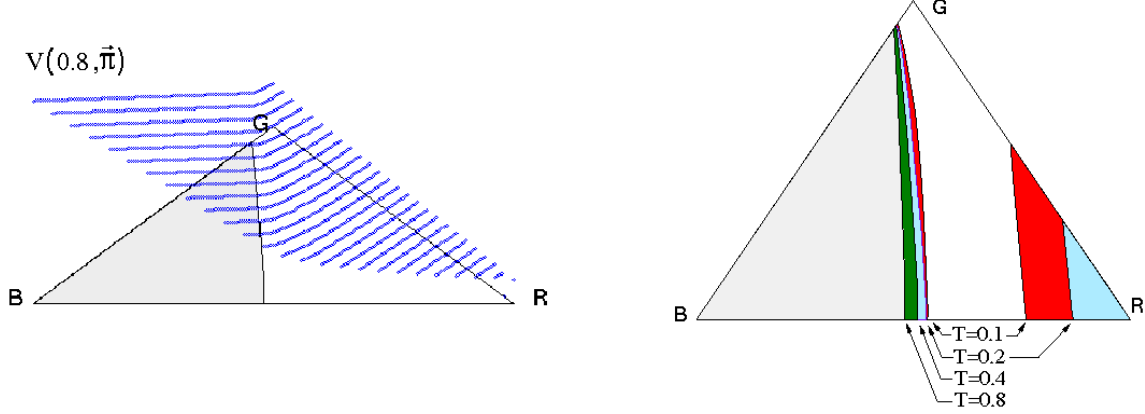


FIGURE 2. Value function and stopping regions of the insurance launch example of Section 6.1. The left panel displays the value function $V(T, \vec{\pi})$, for $\vec{\pi} \in D$ and $T = 0.8$. At $T = 0.8$, if the conditional likelihood process $\vec{\Pi}$ is in the shaded region, the DM stops and selects action $d = 1$. Otherwise, she continues observing until the first time $V(T - t, \vec{\Pi}_t) = H(\vec{\Pi}_t)$. The right panel shows the dependence of the stopping regions on horizon T .

6.2. Bayesian regime detection. Recall the hypothesis testing problem in (1.3). Let $V(\infty, \vec{\pi})$ denote the value function of this *minimization* problem on infinite-horizon. With the notation in (4.12), it is shown in [12] that it is optimal to stop the first time the conditional probability process $\vec{\Pi}$ enters the region $\cup_{k \in E} \Gamma_{\infty, k}$ where $\Gamma_{\infty, k} \triangleq \{\vec{\pi} \in D : V(\infty, \vec{\pi}) = H_k(\vec{\pi})\}$ in terms of the functions $H_k(\vec{\pi}) = \sum_{i \in E} \mu_{k,i} \pi_i$. Each $\Gamma_{\infty, k}$ is a convex region with non-empty interior around k 'th corner of the simplex D . Namely, an observer stops whenever the conditional likelihood of one of the hypotheses is sufficiently high. This structure also extends to the finite-horizon problem. Since $V(\infty, \vec{\pi}) \leq V(T, \vec{\pi})$, we have $\Gamma_{\infty, k} \subseteq \Gamma_{T, k}$, for $k \in E$ and $T < \infty$. In plain words, regardless of the remaining time to maturity, the observer selects immediately one of the hypotheses when the conditional likelihoods process $\vec{\Pi}$ is around the corners of D (i.e., if there is sufficient posterior statistical evidence).

In Figure 3, we illustrate the time-dependence of the solution structure using a simple example with two hypotheses $H_1 : \Lambda = \lambda_1$ and $H_2 : \Lambda = \lambda_2$ on the arrival rate only. The problem in infinite horizon where there are two hypotheses on the arrival rate was solved for the first time by [32] (with $\lambda_2 > \lambda_1$ without loss of generality). The authors showed that the immediate stopping is optimal if and only if $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) \leq \mu_{2,1} + \mu_{1,2}$ (see [32, Theorem 2.1]). Hence the inequality $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) > \mu_{2,1} + \mu_{1,2}$ has to be satisfied in any finite-horizon problem with non-trivial solution.

In Figure 3, under H_1 the arrival rate is $\lambda_1 = 1$ while under H_2 it is $\lambda_2 = 5$. For the Bayes risk given in (1.3), we select $\mu_{1,2} = \mu_{2,1} = 2$ for the penalty costs for selecting the wrong hypothesis. This numerical example corresponds to the one considered in [32, Figures 2-3]. The left panel of Figure 3 shows the value functions $V(T, \cdot)$ with horizons $T = 0.1, T = 0.2, T = 0.4$ and $T = 2$ respectively, and the terminal reward $H(\vec{\pi}) = \min\{\mu_{1,2}\pi_2; \mu_{2,1}(1 - \pi_2)\}$ on the state space of $\pi_2 \in [0, 1]$. We see that as more time is available to make the decision, the value function decreases, as expected. The right panel of Figure 3 shows that the continuation region widens as time to maturity increases. We also observe that the boundary curves approaches the solution structure of problem with infinite horizon. [32] obtain a continuation region of $[0.22, 0.70]$, very close to ours of $[0.230, 0.705]$ for $T > 1$.

Let us define the lower boundary curve $T \mapsto b_1(T) \triangleq \sup\{\pi_2 \in [0, 1] : V(T, \vec{\pi}) = 2\pi_2\}$. Clearly $b_1(0) = 0.5$. In the right panel, we observe that the lower boundary curve $b_1(\cdot)$ has a discontinuity



FIGURE 3. Bayesian regime detection example of Section 6.2. The left panel shows the value functions $V(T, \vec{\pi})$ for various time horizons T . The right panel shows the stopping regions $\Gamma_{T,k}$ (namely $\Gamma_{T,0}$ below the lower curve and $\Gamma_{T,1}$ above the higher curve) for $T = 2$.

at $T = 0$ (jumping from $\pi_2 = 0.5$ to approximately $\pi_2 = 0.25$) and then remaining constant until about $T = 0.2$. Note that the point $\vec{\pi} = (\pi_1, \pi_2) = (0.5, 0.5)$ is the global maximum of the terminal cost function $H(\vec{\pi})$. Starting at the point $(0.5 + \varepsilon, 0.5 - \varepsilon)$, for $\varepsilon \geq 0$ and small, as long as there is no jump, the conditional likelihood process $\vec{\Pi}$ drifts (quickly) toward the point $\vec{\pi} = (\pi_1, \pi_2) = (1, 0)$ and away from this maximum. For very small values of T , the probability of observing a jump is low and thus it is optimal to continue. Therefore, the lower curve in Figure 3 is discontinuous around $T = 0$. The rate of drift of the process $\vec{\Pi}$ to the point $(1, 0)$ decreases as π_2 decreases and approaches the point $(1, 0)$ (see (2.14)). As a result, at points $\vec{\pi}$ where π_2 is small, the effect of waiting cost becomes dominant and it is optimal to stop even if T is small.

The following remark summarizes our discussion on this problem and states that the behavior of the lower boundary curve around $T = 0$ holds for any set of parameters $\lambda_2 > \lambda_1$, $\mu_{1,2}$, $\mu_{2,1}$. Its proof can be found in the Appendix.

Remark 6.1. Consider the hypothesis-testing problem in (1.3) with two simple hypotheses on the arrival rate: $H_1 : \Lambda = \lambda_1$ and $H_2 : \Lambda = \lambda_2$ (with $\lambda_2 > \lambda_1$). The continuation region C_T is non-empty (for $T > 0$) if and only if $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) > \mu_{2,1} + \mu_{1,2}$. The boundary curve $T \mapsto b_1(T) \triangleq \sup\{\pi_2 \in [0, 1] : V(T, \vec{\pi}) = \mu_{1,2}\pi_2\}$ is discontinuous at $T = 0$, and there is an interval around $T = 0$ at which $b_1(\cdot)$ is constant.

6.3. Optimal replacement of a system. Here we consider the reliability problem in (1.4). In this problem, the unobservable Markov process M represents the current productivity of a given machine, and the n 'th state (defective state) of M is absorbing. The objective is to find the best time to replace the equipment in order to maximize the net lifetime earnings. The problem is studied by [24] under certain assumptions on $(q_{i,j})_{i,j \in E}$, $\vec{\lambda}$, $\vec{\mu}$ and \vec{c} such that the infinitesimal look-ahead (ILA) rule $\tau^{ILA} := \inf\{t \geq 0 : \sum_i r_i \Pi_t^{(i)} < 0\}$ is optimal where $r_i \triangleq c_i + \sum_{j \neq i} (\mu_j - \mu_i) q_{i,j}$ (cf. Lemma 4.1). More precisely these assumptions are (i) $q_i \neq 0$ for $i = 1, \dots, n-1$, with $q_n = 0$ (ii) $r_1 \geq r_2 \geq \dots \geq r_n = c_n$, with $c_n < 0$ (iii) $0 < \lambda_1 \leq \dots \leq \lambda_n$, (iv) $q_{in} > \lambda_n! - \lambda_i$ for $i = 1, \dots, n-1$.

It follows as a corollary to [22, Theorem 3.1] that $\tau^{ILA} \wedge T$ is an optimal stopping rule for the finite horizon problem under these assumptions. Therefore, the region $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$ does not depend on T . This occurs because the instantaneous revenue rates r_i 's completely summarize the relative worth of different machine states, and the sum $\sum_{i \in E} r_i \Pi_t^{(i)}$ is monotonically non-increasing over time $\mathbb{P}^{\vec{\pi}}$ -almost surely for all $\vec{\pi} \in D$ (see [24, Theorem 2]). Thus, T only plays a role insofar as allowing the DM to collect profits before the machine deteriorates.

We illustrate this degeneracy in Figure 4. In this example, we select the parameters to fit the framework of [24]. We have a machine that moves through three regimes $E = \{1, 2, 3\} \equiv$

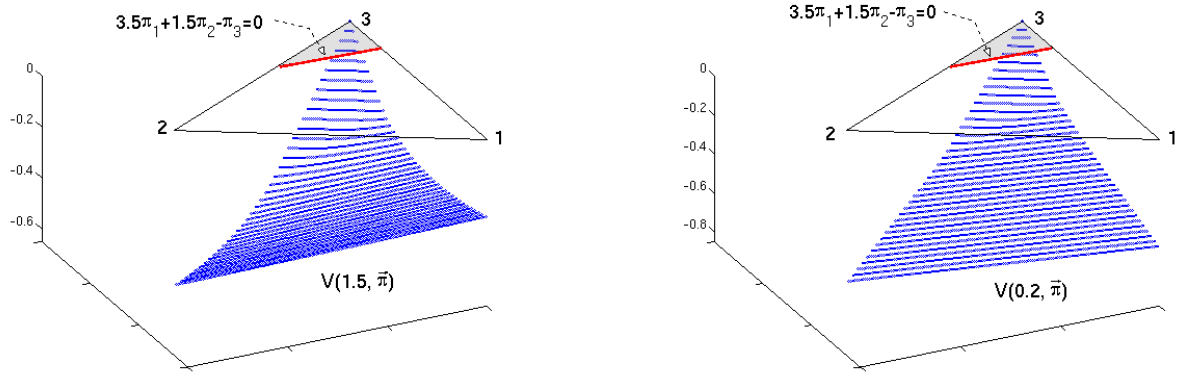


FIGURE 4. Value function $V(T, \vec{\pi})$ of the reliability example of Section 6.3. The shaded regions represent the computed stopping regions $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$. Left panel shows $T = 1.5$, right panel shows $T = 0.2$. The shaded regions are the same in both panels. Note however the different z -scales. The panels also show the line $3.5\pi_1 + 1.5\pi_2 - \pi_3 = 0$, which is the stopping boundary of the ILA rule in (6.1).

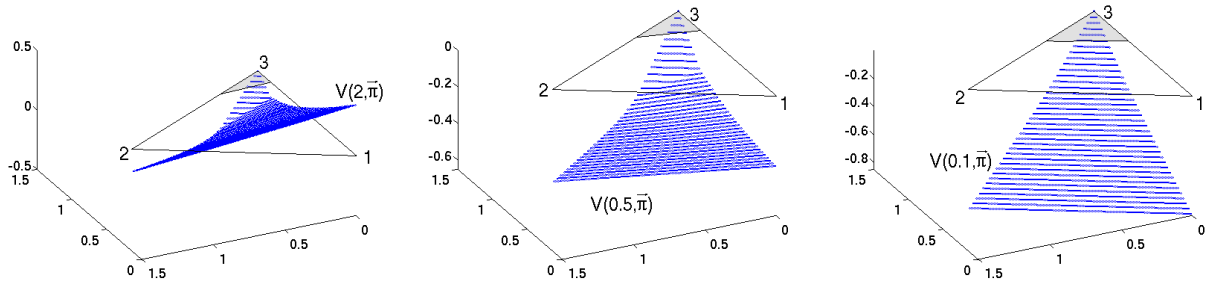


FIGURE 5. The second example for the reliability problem of Section 6.3 with the new parameters in (6.2). In the left panel $T = 2$, in the middle $T = 0.5$, and in the right panel $T = 0.1$. In each picture, the function $V(T, \vec{\pi})$ is plotted on D . The shaded regions are the sets $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$.

$\{\text{Good}, \text{Average}, \text{Poor}\}$ with transition matrix

$$Q = \begin{pmatrix} -4 & 1.5 & 2.5 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & 0 \end{pmatrix}.$$

At different states, the running profit from operating the machine is $\vec{c} = [1, 0, -1]$, and shutting down the machine for maintenance involves a cost of $\vec{\mu} = [-1, -1, 0]$. Thus, it is costly to shutdown a machine until it is in the *Poor* state. In each state, the breakdowns occur according to independent Poisson processes with intensities $\vec{\lambda} = [2, 3, 4]$. In this setting we have $\vec{r} = \{r_1, r_2, r_3\} = \{3.5, 1.5, -1\}$ so that

$$(6.1) \quad \tau^{ILA} = \inf\{t \geq 0 : 3.5\Pi_t^{(1)} + 1.5\Pi_t^{(2)} - \Pi_t^{(3)} < 0\}.$$

The left and right panels of Figure 4 show the functions $V(T, \vec{\pi})$ and the regions $\{\vec{\pi} \in D : (T, \vec{\pi}) \in \Gamma_T\}$ for $T = 1.5$ and $T = 0.2$ respectively. We see that $V(0.2, \vec{\pi}) < V(1.5, \vec{\pi})$ but the regions $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$ for $T = 0.2$ and $T = 1.5$ completely matches the region $\{\vec{\pi} \in D : 3.5\pi_1 + 1.5\pi_2 - \pi_3 \leq 0\}$, at least modulo the D -discretization necessary for numerical implementation.

This degenerate structure would disappear if one removes some of the assumptions in [24], for example the special form of generator Q and/or the arrival rates $\vec{\lambda}$ above. We give an example in Figure 5 where

$$(6.2) \quad Q = \begin{pmatrix} -1 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \vec{\lambda} = [\lambda_1, \lambda_2, \lambda_3] = [1, 4, 7].$$

We keep other parameters the same as in the previous example. In this example, the instantaneous net gain $\sum_{i \in E} r_i \Pi_t^{(i)} = 1.5\Pi_t^{(1)} + 0.5\Pi_t^{(2)} - \Pi_t^{(3)}$ is not monotonically non-increasing $\mathbb{P}^{\vec{\pi}}$ -almost surely for all $\vec{\pi} \in D$ anymore. For example, using (2.14) it can be shown that $d(1.5x_1(t, \vec{\pi}) + 0.5x_2(t, \vec{\pi}) - x_3(t, \vec{\pi}))/dt|_{t=0} > 0$ at the point $\vec{\pi} = (\pi_1, \pi_2, \pi_3) = (0.45, 0.45, 0.1)$. Figure 5 shows that the structure of the stopping region is indeed time dependent. The stopping region expands as time to maturity decreases. Moreover, in this problem the transition rates of M are lower. Therefore, the DM can obtain positive net gain when M starts from the state $\{1\}$ and there is enough time to operate the system. Indeed, the first panel in Figure 5 shows that for $T = 2$ the value function is positive around the corner $\{1\}$.

6.4. Technology adoption example. To illustrate an example for the discrete cost structure of Section 5, we consider an IT company, which is planning to add a new technological feature to its products. The benefit of the technology is unknown, but will improve over time as customer awareness grows and production is streamlined. The company wishes to adopt the technology at the optimal time that best resolves the tension between early adoption (with high production costs) and late adoption (with opportunity costs due to late market entry). A similar setting has been studied recently by [39] and goes all the way back to [31].

Suppose that after T years the technology becomes obsolete and let $M = \{M_t\}_{t \geq 0}$ represent the profitability/value of the technology with state space $E = \{1, 2, 3\} \equiv \{Low, Med, High\}$. The generator of M is

$$Q = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, M sequentially moves through the phases $Low \rightarrow Med \rightarrow High$. The firm may incorporate the feature at the minimal level (action $d = 1$), at the maximum level ($d = 2$), or not at all ($d = 0$). The profit functions are given by

$$\mu_{k,i} = \begin{bmatrix} -1 & 3 & 4 \\ -4 & 2 & 10 \end{bmatrix}, \quad k \in \{1, 2\}, \quad i \in E,$$

with zero profit when $d = 0$.

The observation process X corresponds to competitor contract sales and is represented by a compound Poisson process with mark space $Y_k \in B = \{1, 2\} \equiv \{Large, Small\}$. The M -modulated intensity of X is $\vec{\lambda} = [\lambda_1, \lambda_2, \lambda_3] = [3, 5, 3]$ and the mark distributions on B are $[0.2, 0.8], [0.5, 0.5], [0.8, 0.2]$ respectively. Contracts signed by competitors are opportunity costs and the objective function is of the type (5.1) (with zero discounting $\rho = 0$):

$$V(T, \vec{\pi}) = \sup_{\tau \leq T, d \in \mathcal{F}_\tau^X} \mathbb{E}^{\vec{\pi}} \left[\sum_{j=1}^{N_\tau} K(Y_j) + \sum_{k=1}^2 1_{\{d=k\}} \left(\sum_{i \in E} \mu_{k,i} \cdot 1_{\{M_\tau=i\}} \right) \right],$$

where $T = 1, K(1) = -3, K(2) = -1$.

The triangular regions in Figure 6 are the state space $D = \{\vec{\pi} \in \mathbb{R}_+^3 : \pi_{Low} + \pi_{Med} + \pi_{High} = 1\}$. In the panels, we show how the stopping regions expand as the time to maturity approaches (from left to right) as indicated in Remark 4.2. When $T = 1$, (left panel) we see that if the DM stops, she

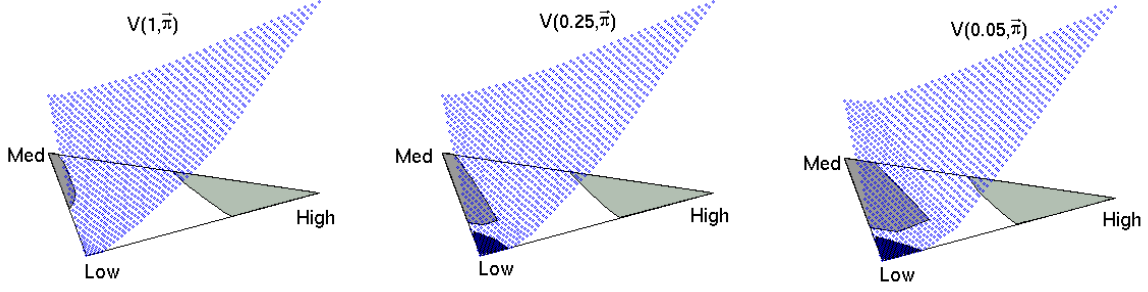


FIGURE 6. Value function $V(T, \vec{\pi})$ of the technology adoption example 6.4 plotted together with the stopping regions (shaded: $d = 2$ lighter color, $d = 1$ darker, $d = 0$ black). Left panel: $T = 1$, middle panel: $T = 0.25$, right panel: $T = 0.05$.

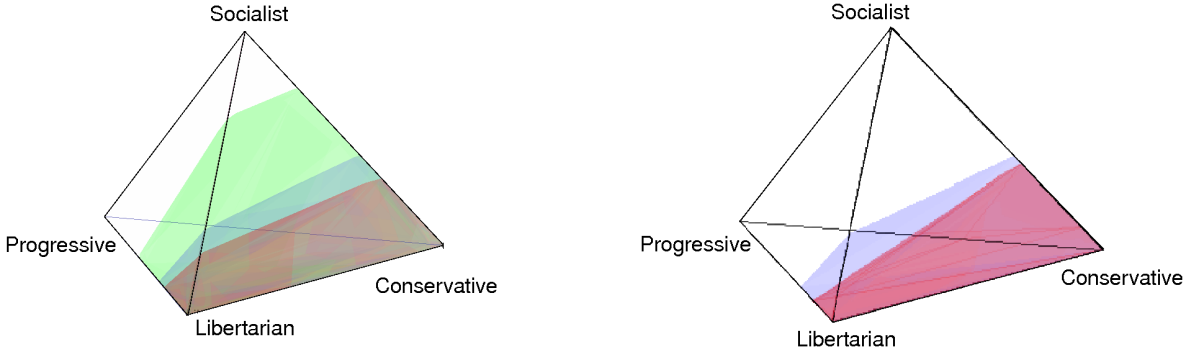


FIGURE 7. Stopping regions $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\} \subset \Gamma_T$ of the targeting example of Section 6.5 for $T = 2$. On the left panel we illustrate the effect of the waiting cost c , with the shaded polyhedra representing stopping regions for $c = -0.1, c = -0.2, c = -0.4$ respectively. On the right panel we take $c = -0.2$, and we display the effect of changing the arrival rate from $\lambda_L = 4$ (blue/lighter stopping region) to $\lambda_L = 10$ (red/darker stopping region).

either selects $d = 1$, or $d = 2$ if there is sufficient evidence that M is at *Med* or *High* respectively. For $T = 1$, the decision $d = 0$ is never considered since the DM can wait for M to move to better states. Note that, if T is small (middle and right panels) and if M seems to be at *Low* state, the DM does not have enough time to wait for M to jump to a new state. By stopping immediately, she at least gets rid of the opportunity costs.

Around the *Med* corner there is high competitor activity ($\lambda_2 = 5$), and this increases in the opportunity costs (given by $K(\cdot)$). As a result the DM always stops, she does not wait for M to move to *High* state. Since the expected reward of minimal commitment is higher than that of maximum commitment around this corner, she selects $d = 1$. The DM selects $d = 2$ only if there is sufficient statistical evidence that the technology has reached its *High* benefit.

6.5. A targeting problem. As a final illustration we present a *targeting* example, where the objective is to maximize the probability of M belonging to some favorable set $B \subseteq E$.

An industrial conglomerate is seeking a business-favorable government legislation and employs a lobbyist for that purpose. The lobbyist maintains government contacts and will try to time her action to maximize the probability of the law passing. Suppose the passage of legislation depends on the current political climate M_t in the country that can be one of the following four states: $E = \{1, 2, 3, 4\} \equiv \{\text{Libertarian}, \text{Conservative}, \text{Progressive}, \text{Socialist}\}$. For simplicity we assume

that the law will pass if the climate is in $B = \{\text{Libertarian}, \text{Progressive}\}$ and fail otherwise. Suppose that the generator of M is

$$Q = \begin{pmatrix} -1 & 0.5 & 0.5 & 0 \\ 0.5 & -1.5 & 0.5 & 0.5 \\ 1 & 0.5 & -2 & 0.5 \\ 0 & 1 & 0.5 & -1.5 \end{pmatrix}.$$

We postulate that the objective function is $\mathbb{E}^{\vec{\pi}}[c\tau] + \mathbb{P}^{\vec{\pi}}(M_\tau \in B)$, where the constant $c \leq 0$ denotes the running cost of maintaining the lobby. Information is obtained via a simple Poisson process counting the passing of other business-friendly legislation, with M -modulated intensities $\vec{\lambda} \equiv [\lambda_L, \lambda_C, \lambda_P, \lambda_S] = [4, 3, 2, 1]$. The time horizon is $T = 2$ years.

Figure 7 shows the stopping regions of this example inside the tetrahedron D . The left panel shows the effect of changing the waiting cost c ; as c increases in absolute value, the DM is more “impatient” and will stop sooner, compare with Remark 4.5. The right panel of Figure 7 shows the effect of increasing λ_L to $\lambda_L = 10$. As intuition suggests, this shrinks the continuation region because the data is now more informative. We see that the continuation region \mathcal{C}_T expands especially around the ‘Libertarian’ corner, as the DM can now be fairly confident in detecting that regime (as it has a much higher arrival intensity).

APPENDIX A1. SAMPLE PATHS OF $\vec{\Pi}$

In this appendix, we prove Lemma (2.1), and we derive the characterization of the sample paths given in (2.10-2.11).

Proof of Lemma 2.1. Let Ξ be a set of the form

$$\Xi = \{N_{t_1} = m_1, \dots, N_{t_k} = m_k; (Y_1, \dots, Y_{m_k}) \in B\}$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_k = t$ with $0 \leq m_1 \leq \dots \leq m_k$ for $k \in \mathbb{N}$, and B is a Borel set in $\mathcal{B}(\mathbb{R}^{m_k})$. Since t_j and m_j ’s are arbitrary, to prove (2.9) it is then sufficient to establish

$$\mathbb{E}^{\vec{\pi}} \left[1_\Xi \cdot \mathbb{P}^{\vec{\pi}} \{M_t = i | \mathcal{F}_t^X\} \right] = \mathbb{E}^{\vec{\pi}} \left[1_\Xi \cdot \frac{L_i^{\vec{\pi}}(t, N_t : (\sigma_k, Y_k), i \leq N_t)}{L^{\vec{\pi}}(t, N_t : (\sigma_k, Y_k), i \leq N_t)} \right].$$

Conditioning on the path of M , the left-hand side (LHS) above equals

$$\begin{aligned} LHS &= \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} \mathbb{P}^{\vec{\pi}} \left\{ N_{t_1} = m_1, \dots, N_{t_k} = m_k; (Y_1, \dots, Y_{m_k}) \in B \mid M_s; s \leq t \right\} \right] \\ &= \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} \int_{B \times \Upsilon(t_1, \dots, t_k)} \mathbb{P}^{\vec{\pi}} \left\{ \sigma_1 \in ds_1, \dots, \sigma_{m_k} \in s_{m_k}; Y_1 \in dy_1, \dots, dY_{m_k} \in dy_{m_k} \mid M_s; s \leq t \right\} \right] \end{aligned}$$

where

$$\Upsilon(t_1, \dots, t_k) = \{s_1, \dots, s_{m_k} \in \mathbb{R}_+^{m_k} : s_1 \leq \dots \leq s_{m_k} \leq t \text{ and } s_{m_j} \leq t_j < s_{m_{j+1}} \text{ for } j = 1, \dots, k\}.$$

Then, by Fubini’s theorem we have

$$\begin{aligned} LHS &= \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} \int_{B \times \Upsilon(t_1, \dots, t_k)} e^{-I(t)} \prod_{l=1}^{m_k} \sum_{j \in E} 1_{\{M_{s_l}=i\}} \lambda_j f_j(y_l) ds_l \nu(dy_l) \right] \\ &= \int_{B \times \Upsilon(t_1, \dots, t_k)} L_i^{\vec{\pi}}(t, m_k : (s_j, y_j), j \leq m_k) \prod_{l=1}^{m_k} ds_l \cdot \nu(dy_l) \\ &= \int_{B \times \Upsilon(t_1, \dots, t_k)} \frac{L_i^{\vec{\pi}}(t, m_k : (s_j, y_j), j \leq m_k)}{L^{\vec{\pi}}(t, m_k : (\sigma_j, Y_j), j \leq m_k)} \cdot L^{\vec{\pi}}(t, m_k : (s_j, y_j), j \leq m_k) \prod_{l=1}^{m_k} ds_l \cdot \nu(dy_l) \end{aligned}$$

Another application of Fubini's theorem gives LHS =

$$\begin{aligned}
& \mathbb{E}^{\vec{\pi}} \left[\sum_{i \in E} 1_{\{M_t=i\}} \int_{B \times \Upsilon(t_1, \dots, t_k)} \frac{L_i^{\vec{\pi}}(t, m_k : (s_j, y_j), j \leq m_k)}{L^{\vec{\pi}}(t, m_k : (\sigma_j, y_j), j \leq m_k)} \cdot e^{-I(t)} \prod_{l=1}^{m_k} \sum_{j \in E} 1_{\{M_{s_l}=i\}} \lambda_j f_j(y_l) \cdot \prod_{l=1}^{m_k} ds_l \cdot \nu(dy_l) \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\sum_{i \in E} 1_{\{M_t=i\}} \mathbb{E}^{\vec{\pi}} \left[1_{\{N_{t_1}=m_1, \dots, N_{t_k}=m_k; (Y_1, \dots, Y_{m_k}) \in B\}} \cdot \frac{L_i^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)}{L^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)} \middle| M_s; s \leq t \right] \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\mathbb{E}^{\vec{\pi}} \left[1_{\{N_{t_1}=m_1, \dots, N_{t_k}=m_k; (Y_1, \dots, Y_{m_k}) \in B\}} \cdot \frac{L_i^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)}{L^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)} \middle| M_s; s \leq t \right] \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[1_{\Xi} \cdot \frac{L_i^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)}{L^{\vec{\pi}}(t, N_t : (\sigma_j, Y_j), j \leq N_t)} \right],
\end{aligned}$$

and this concludes the proof. \square

Proof of Remark 2.1. In order to establish (2.10-2.11), let $\mathbb{E}_j[\cdot]$ denote the expectation operator $\mathbb{E}^{\vec{\pi}}[\cdot | M_0 = j]$, and let $t_m \leq t \leq t+u < t_{m+1}$. Here t_m and t_{m+1} can be considered as the sample realization $\sigma_m(\omega)$ and $\sigma_{m+1}(\omega)$ of the m 'th and $m+1$ 'st arrival times respectively. Using the definition of $L_i^{\vec{\pi}}$ in (2.7) we have $L_i^{\vec{\pi}}(t+u, m : (t_k, y_k), k \leq m) = \sum_{j \in E} \pi_j \cdot \mathbb{E}_j [1_{\{M_{t+u}=i\}} \cdot e^{-I(t+u)} \cdot \prod_{k=1}^m \ell(t_k, y_k)]$

$$\begin{aligned}
&= \sum_{j \in E} \pi_j \cdot \mathbb{E}_j \left[\mathbb{E}_j \left[1_{\{M_{t+u}=i\}} \cdot e^{-I(t+u)} \cdot \prod_{k=1}^m \ell(t_k, y_k) \middle| M_s : s \leq t \right] \right] \\
&= \sum_{j \in E} \pi_j \cdot \mathbb{E}_j \left[e^{-I(t)} \left(\prod_{k=1}^m \ell(t_k, y_k) \right) \mathbb{E}_j \left[1_{\{M_{t+u}=i\}} \cdot e^{-(I(t+u)-I(t))} \middle| M_s : s \leq t \right] \right].
\end{aligned} \tag{A1.1}$$

Using the Markov property of M , the last expression in (A1.1) can be written as

$$\begin{aligned}
&= \sum_{j \in E} \pi_j \cdot \mathbb{E}_j \left[e^{-I(t)} \left(\prod_{k=1}^m \ell(t_k, y_k) \right) \cdot \sum_{l \in E} 1_{\{M_t=l\}} \cdot \mathbb{E}_l [1_{\{M_u=i\}} \cdot e^{-I(u)}] \right] \\
&= \sum_{l \in E} \mathbb{E}_l [1_{\{M_u=i\}} e^{-I(u)}] \cdot \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=l\}} \cdot e^{-I(t)} \prod_{k=1}^m \ell(t_k, y_k) \right] \\
&= \sum_{l \in E} \mathbb{E}_l [1_{\{M_u=i\}} e^{-I(u)}] \cdot L_l^{\vec{\pi}}(t, m : (\sigma_k, y_k), k \leq m).
\end{aligned}$$

Then the explicit form of $\vec{\Pi}$ in (2.9) implies that for $\sigma_m \leq t \leq t+u < \sigma_{m+1}$, we have

$$\begin{aligned}
\text{(A1.2)} \quad \Pi_i(t+u) &= \frac{\sum_{l \in E} L_l^{\vec{\pi}}(t, m : (\sigma_k, y_k), k \leq m) \cdot \mathbb{E}_l [1_{\{M_u=i\}} \cdot e^{-I(u)}]}{\sum_{j \in E} \sum_{l \in E} L_l^{\vec{\pi}}(t, m : (\sigma_k, y_k), k \leq m) \cdot \mathbb{E}_l [1_{\{M_u=j\}} e^{-I(u)}]} \\
&= \frac{\sum_{l \in E} \Pi_l(t) \cdot \mathbb{E}_l [1_{\{M_u=i\}} e^{-I(u)}]}{\sum_{j \in E} \sum_{l \in E} \Pi_l(t) \cdot \mathbb{E}_l [1_{\{M_u=j\}} e^{-I(u)}]} = \frac{\mathbb{E}^{\vec{\Pi}_t} [1_{\{M_u=i\}} e^{-I(u)}]}{\sum_{j \in E} \mathbb{E}^{\vec{\Pi}_t} [1_{\{M_u=j\}} e^{-I(u)}]} = \frac{\mathbb{P}^{\vec{\pi}}\{\sigma_1 > u, M_u = i\}}{\mathbb{P}^{\vec{\pi}}\{\sigma_1 > u\}} \bigg|_{\vec{\pi}=\vec{\Pi}_t}.
\end{aligned}$$

On the other hand, the expression in (2.7) gives

(A1.3)

$$\begin{aligned} L_i^{\vec{\pi}}(\sigma_{m+1}, m+1 : (\sigma_k, Y_k), k \leq m+1) &= \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} e^{-I(t)} \prod_{k=1}^{m+1} \ell(t_k, y_k) \right] \Bigg|_{\substack{t=\sigma_{m+1} \\ (t_k=\sigma_k, y_k=Y_k)_{k \leq m+1}}} \\ &= \lambda_i f_i(Y_{m+1}) \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} e^{-I(t)} \prod_{k=1}^m \ell(t_k, y_k) \right] \Bigg|_{\substack{t=\sigma_{m+1} \\ (t_k=\sigma_k, y_k=Y_k)_{k \leq m}}}. \end{aligned}$$

Observe that for fixed time t , we have $M_t = M_{t-}$, $\mathbb{P}^{\vec{\pi}}$ -a.s. and $L_i^{\vec{\pi}}(t, m : (t_k, y_k), k \leq m) = L_i^{\vec{\pi}}(t-, m : (t_k, y_k), k \leq m)$ when $t_m < t$. Then we have

$$L_i^{\vec{\pi}}(\sigma_{m+1}, m+1 : (\sigma_k, Y_k), k \leq m+1) = \lambda_i f_i(Y_{m+1}) \cdot L_i^{\vec{\pi}}(\sigma_{m+1}-, m : (\sigma_k, Y_k), k \leq m),$$

due to (A1.3). Hence, at arrival times $\sigma_1, \sigma_2, \dots$ of X , the process $\vec{\Pi}$ exhibits a jump behavior and satisfies the recursive relation $\Pi_i(\sigma_{m+1}) =$

$$(A1.4) \quad \frac{\lambda_i f_i(Y_{m+1}) L_i^{\vec{\pi}}(\sigma_{m+1}-, m : (\sigma_k, Y_k), i \leq m)}{\sum_{j \in E} \lambda_j f_j(Y_{m+1}) L_j^{\vec{\pi}}(\sigma_{m+1}-, m : (\sigma_k, Y_k), k \leq m)} = \frac{\lambda_i f_i(Y_{m+1}) \Pi_i(\sigma_{m+1}-)}{\sum_{j \in E} \lambda_j f_j(Y_{m+1}) \Pi_j(\sigma_{m+1}-)}$$

for $m \in \mathbb{N}$.

The identities in (A1.2) and (A1.4) give (2.10-2.11). By repeating (A1.1-A1.2) with $m = 0$ (i.e., with no arrivals on $[0, t+s]$), we see that the paths $t \mapsto \vec{x}(t, \vec{\pi})$ have the semigroup property $\vec{x}(t+u, \vec{\pi}) = \vec{x}(u, \vec{x}(t, \vec{\pi}))$. \square

APPENDIX A2. SUPPLEMENTARY RESULTS AND OTHER PROOFS

Proof of Proposition 3.1. The inequality $V_m(s, \vec{\pi}) \leq V(s, \vec{\pi})$ is immediate. To show the second inequality, let τ be an \mathbb{F} -stopping time less than s \mathbb{P} -a.s.. Then we have

$$\begin{aligned} (A2.1) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right] &= \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right. \\ &\quad \left. + 1_{\{\tau > \sigma_m\}} \left[\int_{\sigma_m}^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) - e^{-\rho \sigma_m} H(\vec{\Pi}_{\sigma_m}) \right] \right] \\ &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right. \\ &\quad \left. + 1_{\{\tau > \sigma_m\}} e^{-\rho \sigma_m} \left[\|C\| \int_0^{T-\sigma_m} e^{-\rho t} dt + e^{-\rho(\tau-\sigma_m)} H(\vec{\Pi}_\tau) - H(\vec{\Pi}_{\sigma_m}) \right] \right] \\ &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right] + (T\|C\| + 2\|H\|) \cdot \mathbb{E}^{\vec{\pi}} [e^{-\rho \sigma_m} 1_{\{T > \sigma_m\}}] \end{aligned}$$

where the last line follows since $\tau \leq s \leq T$ and $\{\tau > \sigma_m\} \subseteq \{T > \sigma_m\}$. Using the Cauchy-Schwarz inequality and the inequalities $\mathbb{P}^{\vec{\pi}}\{T > \sigma_m\} \leq \mathbb{E}^{\vec{\pi}}[1_{\{T > \sigma_m\}}(T/\sigma_m)] \leq T \cdot \mathbb{E}^{\vec{\pi}}[1/\sigma_m]$ we obtain

$$\begin{aligned} (A2.2) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right] &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right] \\ &\quad + (T\|C\| + 2\|H\|) \sqrt{T \mathbb{E}^{\vec{\pi}}[1/\sigma_m] \mathbb{E}^{\vec{\pi}}[e^{-2\rho \sigma_m}]}. \end{aligned}$$

Note that given M , we have $\mathbb{P}^{\vec{\pi}}[\sigma_1 > t | M] = e^{-I(t)}$, where $I(\cdot)$ is defined as in (2.8). This implies $\mathbb{E}^{\vec{\pi}}[e^{-u\sigma_1} | M] = \mathbb{E}^{\vec{\pi}}\left[\int_{\sigma_1}^{\infty} u \cdot e^{-ut} dt | M\right] = \int_0^{\infty} \mathbb{P}^{\vec{\pi}}[\sigma_1 \leq t | M] u \cdot e^{-ut} dt =$

$$\int_0^{\infty} [1 - e^{-I(t)}] u \cdot e^{-ut} dt \leq \int_0^{\infty} [1 - e^{-\bar{\lambda}t}] u \cdot e^{-ut} dt = \frac{\bar{\lambda}}{u + \bar{\lambda}}.$$

The process X has independent increments conditioned on M . Then, the inequality $\mathbb{E}^{\vec{\pi}}[e^{-u\sigma_m} | M] \leq \left(\frac{\bar{\lambda}}{u + \bar{\lambda}}\right)^m$ follows by induction and we have

$$(A2.3) \quad \mathbb{E}^{\vec{\pi}}[e^{-u\sigma_m}] \leq \left(\frac{\bar{\lambda}}{u + \bar{\lambda}}\right)^m,$$

for all $m \in \mathbb{N}$. Moreover, since $1/\sigma_m = \int_0^{\infty} e^{-\sigma_m u} du$, the inequality in (A2.3) gives $\mathbb{E}^{\vec{\pi}}[1/\sigma_m] \leq \int_0^{\infty} (\bar{\lambda}^m / u + \bar{\lambda})^m du = \bar{\lambda}/(m-1)$, for $m \geq 2$. By using this upper bound in (A2.2) and taking the supremum of both sides we obtain (3.3). \square

Proof of Lemma 3.2. Boundedness and monotonicity are immediate by the definition of the operator J in (3.6). To establish the convexity, we will show that expression in (3.7) is convex (in $\vec{\pi}$) for each t and s .

We first note that $\mathbb{E}^{\vec{\pi}}[e^{-I(t)}] = \sum_{j \in E} \pi_j \mathbb{E}_j[e^{-I(t)}]$ and $m_i(t, \vec{\pi}) = \sum_{j \in E} \pi_j \mathbb{E}_j[1_{\{M_t=i\}} e^{-I(t)}]$ are linear in $\vec{\pi}$ where $m_i(t, \vec{\pi})$ is defined in (2.13) for $i \in E$ and \mathbb{E}_j is the expectation operator $\mathbb{E}[\cdot | M_0 = j]$ for $j \in E$. Then we see that the expression $\mathbb{E}^{\vec{\pi}}[e^{-I(t)}] e^{-\rho t} H(\vec{x}(t, \vec{\pi})) = \max_{k \in \mathcal{A}} e^{-\rho t} \sum_{i \in E} \mu_{k,i} m_i(t, \vec{\pi})$ is convex as the upper envelope of convex functions. Next we let $\vec{\pi} \mapsto w(s, \vec{\pi})$ be a convex mapping for each $s \geq 0$. Then we have $w(s, \vec{\pi}) = \sup_{k \in K_s} \beta_{k,0}(s) + \beta_{k,1}(s)\pi_1 + \dots + \beta_{k,n}(s)\pi_n$, for some index set K_s , and each $\beta_{k,i}(s)$ is a function in s . Using this characterization with the definition of the operator S_i in (3.8) we obtain $\int_0^t e^{-\rho u} \sum_{i \in E} \mathbb{E}^{\vec{\pi}}[1_{\{M_u=i\}} e^{-I(u)}] \cdot \lambda_i S_i w(s-u, \vec{x}(u, \vec{\pi})) du = \int_0^t e^{-\rho u} \sum_{i \in E} \lambda_i m_i(u, \vec{\pi}) \cdot$

$$\begin{aligned} & \left[\int_{\mathbb{R}^d} \sup_{k \in K_{s-u}} \left(\beta_{k,0}(s-u) + \sum_{j \in E} \beta_{k,j}(s-u) \frac{\lambda_j f_j(y) m_j(u, \vec{\pi})}{\sum_{l \in E} \lambda_l f_l(y) m_l(u, \vec{\pi})} \right) f_i(y) \nu(dy) \right] du \\ &= \int_0^t e^{-\rho u} \left[\int_{\mathbb{R}^d} \sup_{k \in K_{s-u}} \left(\sum_{j \in E} [\beta_{k,j}(s-u) + \beta_{k,0}(s-u)] \lambda_j f_j(y) m_j(u, \vec{\pi}) \right) \nu(dy) \right] du. \end{aligned}$$

Since the expression inside the supremum operator are linear in π , the integrand in the inner integral is convex, and therefore so is the expression above. Also note that $\int_0^t e^{-\rho u} \sum_{i \in E} m_i(u, \vec{\pi}) C(\vec{x}(u, \vec{\pi})) du = \int_0^t e^{-\rho u} \sum_{i \in E} c_i m_i(u, \vec{\pi}) du$, where both the integrand and the integral are linear in $\vec{\pi}$. Finally, as the sum of three convex functions $\vec{\pi} \mapsto Jw(t, s, \vec{\pi})$ is convex. Since $J_0 w(s, \vec{\pi})$ is the supremum of convex functions, it is again convex. \square

Proof of Lemma 3.3. Let us define $\Upsilon_T \triangleq \{(t, s) \in \mathbb{R}_+^2 : 0 \leq t \leq s, s \leq T\}$. Then the mapping $(t, s, \vec{\pi}) \mapsto \mathbb{E}^{\vec{\pi}}[e^{-I(t)}] \cdot e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi})) = \left(\sum_{j \in E} \pi_j \mathbb{E}_j[e^{-I(t)}]\right) e^{-\rho t} \cdot H(\vec{x}(t, \vec{\pi}))$ is continuous on the compact set $\Upsilon_T \times D$ due to bounded convergence theorem, the continuity of $H(\cdot)$, and regularity of paths $t \mapsto \vec{x}(t, \vec{\pi})$.

For a (bounded) continuous function $w(\cdot, \cdot)$ on $[0, T] \times D$, the function $S_i w(\cdot, \cdot)$ is again continuous for $i \in E$ due to bounded convergence theorem. Next let $(t_m, s_m, \vec{\pi}_m)_{m \in \mathbb{N}}$ be a sequence converging to a point $(t, s, \vec{\pi}) \in \Upsilon_T \times D$, and let us denote $F_i(u, s, \vec{\pi}) \triangleq C(\vec{x}(u, \vec{\pi})) + \lambda_i S_i w(s-u, \vec{x}(u, \vec{\pi}))$ for

typographical convenience. Then

$$\begin{aligned}
& \left| \int_0^t e^{-\rho u} \sum_{i \in E} \mathbb{E}^{\vec{\pi}} \left[1_{\{M_u=i\}} e^{-I(u)} \right] F_i(u, s, \vec{\pi}) du - \int_0^{t_m} e^{-\rho u} \sum_{i \in E} \mathbb{E}^{\vec{\pi}_m} \left[1_{\{M_u=i\}} e^{-I(u)} \right] F_i(u, s_m, \vec{\pi}_m) du \right| \\
& \leq \left| \int_{t_m}^t e^{-\rho u} \sum_{i \in E} \mathbb{E}^{\vec{\pi}} \left[1_{\{M_u=i\}} e^{-I(u)} \right] F_i(u, s, \vec{\pi}) du \right| \\
& \quad + \left| \int_0^{t_m} e^{-\rho u} \cdot \sum_{i \in E} \left(\mathbb{E}^{\vec{\pi}} \left[1_{\{M_u=i\}} e^{-I(u)} \right] F_i(u, s, \vec{\pi}) - \mathbb{E}^{\vec{\pi}_m} \left[1_{\{M_u=i\}} e^{-I(u)} \right] F_i(u, s_m, \vec{\pi}_m) \right) du \right| \\
& \leq (\|C\| + \bar{\lambda}\|w\|) \int_{t_m}^t e^{-\rho u} du + \int_0^T e^{-\rho u} \sum_{i \in E} \left| \left(\dots - \dots \right) \right| du.
\end{aligned}$$

Note that as $m \rightarrow \infty$, the second integrand above goes to 0, and the whole expression vanishes due to dominated convergence theorem. Hence, we conclude that $Jw(t, s, \vec{\pi})$ in (3.7) is continuous on $\Upsilon_T \times D$. Since this last set is compact, it follows that $Jw(t, s, \vec{\pi})$ is uniformly continuous and $(s, \vec{\pi}) \mapsto J_0 w(s, \vec{\pi}) = \sup_{t \leq s} J_0 w(t, s, \vec{\pi})$ is continuous on $[0, T] \times D$. \square

To prove Proposition 3.2, we first establish the following intermediate result.

Proposition A2.1. *For every $\varepsilon \geq 0$, let us define*

$$(A2.4) \quad r_m^\varepsilon(s, \vec{\pi}) \triangleq \inf\{t \in [0, s] : Jv_m(t, s, \vec{\pi}) \geq J_0 v_m(s, \vec{\pi}) - \varepsilon\}, \quad \vec{\pi} \in D,$$

$$S_1^\varepsilon(s, \vec{\pi}) \triangleq r_0^\varepsilon(s, \vec{\pi}) \wedge \sigma_1 \quad \text{and} \quad S_{m+1}^\varepsilon(s, \vec{\pi}) \triangleq \begin{cases} r_m^{\varepsilon/2}(s, \vec{\pi}) & \text{if } \sigma_1 > r_m^{\varepsilon/2}(s, \vec{\pi}), \\ \sigma_1 + S_m^{\varepsilon/2}(s - \sigma_1, \vec{\Pi}_{\sigma_1}) & \text{if } \sigma_1 \leq r_m^{\varepsilon/2}(s, \vec{\pi}). \end{cases}$$

Then, for every $m \geq 1$ we have

$$(A2.5) \quad \mathbb{E}^{\vec{\pi}} \left[\int_0^{S_m^\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot S_m^\varepsilon(s, \vec{\pi})} H \left(\vec{\Pi}_{S_m^\varepsilon(s, \vec{\pi})} \right) \right] \geq v_m(s, \vec{\pi}) - \varepsilon.$$

Proof. We will prove (A2.5) by an induction on $m \in \mathbb{N}$. For $m = 1$, thanks to (3.6) and (A2.4) the left-hand-side of (A2.5) equals $\mathbb{E} \left[\int_0^{r_0^\varepsilon(s, \vec{\Pi}_0) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot r_0^\varepsilon(s, \vec{\Pi}_0) \wedge \sigma_1} H \left(\vec{\Pi}_{r_0^\varepsilon(s, \vec{\Pi}_0) \wedge \sigma_1} \right) \right]$ $JH(r_0^\varepsilon(s, \vec{\pi}), s, \vec{\pi}) \equiv Jv_0(r_0^\varepsilon(s, \vec{\pi}), s, \vec{\pi}) \geq v_1(s, \vec{\pi}) - \varepsilon$, which proves (A2.5) for $m = 1$.

Now, let us suppose (A2.5) holds for $\varepsilon \geq 0$, and for some $m > 1$, and let us prove that it also holds when m is replaced by $m + 1$. Since $S_{m+1}^\varepsilon(s, \vec{\pi}) \wedge \sigma_1 = r_m^\varepsilon(s, \vec{\Pi}_0) \wedge \sigma_1$, we have

$$\begin{aligned}
& \mathbb{E}^{\vec{\pi}} \left[\int_0^{S_{m+1}^\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot S_{m+1}^\varepsilon(s, \vec{\pi})} \cdot H \left(\vec{\Pi}_{S_{m+1}^\varepsilon(s, \vec{\pi})} \right) \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\int_0^{S_{m+1}^\varepsilon(s, \vec{\pi}) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{S_{m+1}^\varepsilon(s, \vec{\pi}) < \sigma_1\}} e^{-\rho \cdot S_{m+1}^\varepsilon(s, \vec{\pi})} H \left(\vec{\Pi}_{S_{m+1}^\varepsilon(s, \vec{\pi})} \right) \right. \\
&\quad \left. + 1_{\{S_{m+1}^\varepsilon(s, \vec{\pi}) \geq \sigma_1\}} \left[\int_{\sigma_1}^{S_{m+1}^\varepsilon(s, \vec{\pi}) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot S_{m+1}^\varepsilon(s, \vec{\pi})} H \left(\vec{\Pi}_{S_{m+1}^\varepsilon(s, \vec{\pi})} \right) \right] \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\int_0^{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) < \sigma_1\}} H \left(\vec{\Pi}_{r_m^{\varepsilon/2}(s, \vec{\Pi}_0)} \right) + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \geq \sigma_1\}} \cdot \right. \\
&\quad \left. \left[\int_{\sigma_1}^{\sigma_1 + S_m^{\varepsilon/2/2}(s - \sigma_1, \vec{\Pi}_{\sigma_1})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot (\sigma_1 + S_m^{\varepsilon/2/2}(s - \sigma_1, \vec{\Pi}_{\sigma_1}))} H \left(\vec{\Pi}_{\sigma_1 + S_m^{\varepsilon/2/2}(s - \sigma_1, \vec{\Pi}_{\sigma_1})} \right) \right] \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\int_0^{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) < \sigma_1\}} H \left(\vec{\Pi}_{r_m^{\varepsilon/2}(s, \vec{\Pi}_0)} \right) + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \geq \sigma_1\}} e^{-\rho \sigma_1} f_m(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right]
\end{aligned}$$

where the last line follows from the strong Markov property and where

$$f_m(u, \vec{\pi}) = \mathbb{E}^{\vec{\pi}} \left[\int_0^{S_m^{\varepsilon/2}(u, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot S_m^{\varepsilon/2}(u, \vec{\pi})} \cdot H \left(\vec{\Pi}_{S_m^{\varepsilon/2}(u, \vec{\pi})} \right) \right] \geq v_m(u, \vec{\pi}) - \varepsilon/2.$$

The inequality above follows from the induction hypothesis. Then we obtain

$$\begin{aligned}
& \mathbb{E}^{\vec{\pi}} \left[\int_0^{S_{m+1}^\varepsilon(s, \vec{\pi})} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot S_{m+1}^\varepsilon(s, \vec{\pi})} \cdot H \left(\vec{\Pi}_{S_{m+1}^\varepsilon(s, \vec{\pi})} \right) \right] \geq \mathbb{E}^{\vec{\pi}} \left[\int_0^{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt \right. \\
&\quad \left. + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) < \sigma_1\}} H \left(\vec{\Pi}_{r_m^{\varepsilon/2}(s, \vec{\Pi}_0)} \right) + 1_{\{r_m^{\varepsilon/2}(s, \vec{\Pi}_0) \geq \sigma_1\}} e^{-\rho \sigma_1} \cdot v_m(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right] - \frac{\varepsilon}{2} \\
&= J v_m(r_m^{\varepsilon/2}(\vec{\pi}), s, \vec{\pi}) - \frac{\varepsilon}{2} \geq v_{m+1}(\vec{\pi}) - \varepsilon. \text{ Here the equality follows from the definition of the operator } J \text{ in (3.6) and the second equality follows from (A2.4). This concludes the proof of (A2.5). } \square
\end{aligned}$$

Proof of Proposition 3.2. The inequality $V_m \geq v_m$ follows from (A2.5) since $S_m^\varepsilon(s, \vec{\pi}) \leq s \wedge \sigma_m$ by construction. To prove the reverse inequality $V_m \leq v_m$ we will show

$$(A2.6) \quad \mathbb{E} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot \tau \wedge \sigma_m} \cdot H \left(\vec{\Pi}_{\tau \wedge \sigma_m} \right) \right] \leq v_m(s, \vec{\pi}),$$

for every bounded stopping time $\tau \leq s$ and $m \in \mathbb{N}$, by showing

$$\begin{aligned}
(A2.7) \quad & \mathbb{E} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot \tau \wedge \sigma_m} \cdot H \left(\vec{\Pi}_{\tau \wedge \sigma_m} \right) \right] \leq \mathbb{E} \left[\int_0^{\tau \wedge \sigma_{m-k+1}} e^{-\rho t} C(\vec{\Pi}_t) dt \right. \\
& \left. + 1_{\{\tau \geq \sigma_{m-k+1}\}} e^{-\rho \cdot \sigma_{m-k+1}} v_{k-1} \left(s - \sigma_{m-k+1}, \vec{\Pi}_{\sigma_{m-k+1}} \right) + 1_{\{\tau < \sigma_{m-k+1}\}} e^{-\rho \cdot \tau} \cdot H \left(\vec{\Pi}_\tau \right) \right] =: RHS_{k-1},
\end{aligned}$$

for $k = 1, \dots, m + 1$. The inequality (A2.6) will then follow from (A2.7) by taking $k = m + 1$. For $k = 1$, (A2.7) is satisfied as an equality since $v_0(s, \cdot) = H(\cdot)$, for all $s \in [0, T]$. Now, let us assume (A2.7) holds for some $1 \leq k < m + 1$, and let us prove that it also holds for $k + 1$.

Note that RHS_{k-1} in (A2.7) can be written as $RHS_{k-1} = RHS_{k-1}^{(1)} + RHS_{k-1}^{(2)}$, in terms of

$$\begin{aligned} RHS_{k-1}^{(1)} &\triangleq \mathbb{E} \left[\int_0^{\tau \wedge \sigma_{m-k}} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\tau < \sigma_{m-k}\}} e^{-\rho \cdot \tau} \cdot H(\vec{\Pi}_\tau) \right], \\ RHS_{k-1}^{(2)} &\triangleq \mathbb{E} \left[1_{\{\tau \geq \sigma_{m-k}\}} \cdot \left(\int_{\sigma_{m-k}}^{\tau \wedge \sigma_{m-k+1}} e^{-\rho t} C(\vec{\Pi}_t) dt \right. \right. \\ &\quad \left. \left. + 1_{\{\tau \geq \sigma_{m-k+1}\}} e^{-\rho \cdot \sigma_{m-k+1}} \cdot v_{k-1} \left(s - \sigma_{m-k+1}, \vec{\Pi}_{\sigma_{m-k+1}} \right) + 1_{\{\tau < \sigma_{m-k+1}\}} e^{-\rho \cdot \tau} \cdot H(\vec{\Pi}_\tau) \right) \right]. \end{aligned}$$

Lemma 3.1 implies that there exists an $\mathcal{F}_{\sigma_{m-k}}^X$ -measurable random variable R_{m-k} such that

$$\tau \wedge \sigma_{m-k+1} = (\sigma_{m-k} + R_{m-k}) \wedge \sigma_{m-k+1} \quad \text{on } \{\tau \geq \sigma_{m-k}\}.$$

Moreover since $\tau \leq s$, we have $R_{m-k} \leq s - \sigma_{m-k}$ on $\{\tau \geq \sigma_{m-k}\}$. Then we obtain $RHS_{k-1}^{(2)} =$

$$\begin{aligned} \mathbb{E} \left[1_{\{\tau \geq \sigma_{m-k}\}} \cdot \left(\int_{\sigma_{m-k}}^{(\sigma_{m-k} + R_{m-k}) \wedge \sigma_{m-k+1}} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\tau \geq \sigma_{m-k+1}\}} e^{-\rho \sigma_{m-k+1}} v_{k-1} \left(s - \sigma_{m-k+1}, \vec{\Pi}_{\sigma_{m-k+1}} \right) \right. \right. \\ \left. \left. + 1_{\{\sigma_{m-k} + R_{m-k} < \sigma_{m-k+1}\}} e^{-\rho(\sigma_{m-k} + R_{m-k})} \cdot H(\vec{\Pi}_{\sigma_{m-k} + R_{m-k}}) \right) \right]. \end{aligned}$$

Due to strong Markov property, the last expression can be written as

$$(A2.8) \quad RHS_{k-1}^{(2)} = \mathbb{E} \left[1_{\{\tau \geq \sigma_{m-k}\}} \cdot e^{-\rho \cdot \sigma_{m-k}} g_{k-1} \left(R_{m-k}, s - \sigma_{m-k}, \vec{\Pi}(\sigma_{m-k}) \right) \right],$$

where $g_{k-1}(r, u, \vec{\pi}) \triangleq$

$$\mathbb{E} \left[\int_0^{r \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{r < \sigma_1\}} e^{-\rho r} H(\vec{\Pi}(r)) + 1_{\{r \geq \sigma_1\}} e^{-\rho \sigma_1} v_{k-1} \left(u - \sigma_1, \vec{\Pi}(\sigma_1) \right) \right],$$

for $r \leq u$. Then, using the definition of the operator J in (3.6) we have

$$g_{k-1}(r, u, \vec{\pi}) = Jv_{k-1}(r, u, \vec{\pi}) \leq J_0v_{k-1}(u, \vec{\pi}) = v_k(u, \vec{\pi}).$$

As a result, we obtain $RHS_{k-1}^{(2)} \leq \mathbb{E} \left[1_{\{\tau \geq \sigma_{m-k}\}} e^{-\rho \cdot \sigma_{m-k}} v_k \left(u - \sigma_{m-k}, \vec{\Pi}(\sigma_1) \right) \right]$, and this further implies

$$\begin{aligned} (A2.9) \quad &\mathbb{E} \left[\int_0^{\tau \wedge \sigma_m} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \cdot \tau \wedge \sigma_m} H(\vec{\Pi}_{\tau \wedge \sigma_m}) \right] \\ &\leq RHS_{k-1} = \mathbb{E} \left[\int_0^{\tau \wedge \sigma_{m-k}} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\tau < \sigma_{m-k}\}} e^{-\rho \cdot \tau} \cdot H(\vec{\Pi}_\tau) \right] + RHS_{k-1}^{(2)} \\ &\leq \mathbb{E} \left[\int_0^{\tau \wedge \sigma_{m-k}} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\tau < \sigma_{m-k}\}} e^{-\rho \cdot \tau} H(\vec{\Pi}_\tau) + 1_{\{\tau \geq \sigma_{m-k}\}} \cdot e^{-\rho \cdot \sigma_{m-k}} v_k \left(u - \sigma_{m-k}, \vec{\Pi}_{\sigma_1} \right) \right]. \end{aligned}$$

Since the last term equals RHS_k , this completes the proof of (A2.7) by induction. Equation (A2.6) follows when we set $k = m + 1$. Finally, taking the infimum of both sides in (A2.6), we arrive at the desired inequality $V_m \leq v_m$. \square

Proof of Lemma 3.5. Using the definition of the operator J in (3.6) we obtain

$$\begin{aligned}
Jw(t, s, \vec{\pi}) &= \mathbb{E}^{\vec{\pi}} \left[\int_0^{t \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{t < \sigma_1\}} \cdot e^{-\rho t} H(\vec{\Pi}_t) + 1_{\{\sigma_1 \leq t\}} \cdot e^{-\rho \sigma_1} w(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right] \\
&= \mathbb{E}^{\vec{\pi}} \left[\int_0^{u \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt + \int_{u \wedge \sigma_1}^{t \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt - 1_{\{u < \sigma_1\}} \cdot e^{-\rho u} H(\vec{\Pi}_u) + 1_{\{u < \sigma_1\}} \cdot e^{-\rho u} H(\vec{\Pi}_u) \right. \\
&\quad \left. + 1_{\{t < \sigma_1\}} \cdot e^{-\rho t} H(\vec{\Pi}_t) + 1_{\{\sigma_1 \leq u\}} e^{-\rho \sigma_1} w(s - \sigma_1, \vec{\Pi}_{\sigma_1}) + 1_{\{u < \sigma_1 \leq t\}} \cdot e^{-\rho \sigma_1} w(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right] \\
&= Jw(u, s, \vec{\pi}) + \mathbb{E}^{\vec{\pi}} \left[-1_{\{\sigma_1 > u\}} \cdot e^{-\rho u} H(\vec{\Pi}_u) + 1_{\{\sigma_1 > u\}} \left(\int_u^{t \wedge \sigma_1} e^{-\rho t} C(\vec{\Pi}_t) dt \right) \right. \\
&\quad \left. + 1_{\{\sigma_1 > u\}} \left(1_{\{\sigma_1 > t\}} \cdot e^{-\rho t} H(\vec{\Pi}_t) + 1_{\{\sigma_1 \leq t\}} \cdot e^{-\rho \sigma_1} \cdot w(s - \sigma_1, \vec{\Pi}_{\sigma_1}) \right) \right].
\end{aligned}$$

On $\{\sigma_1 > u\}$, we have $\sigma_1 \wedge t = u + (\sigma_1 \wedge (t - u)) \circ \theta_u$. Then the Markov property of $\vec{\Pi}$ gives

$$\begin{aligned}
Jw(t, s, \vec{\pi}) &= Jw(u, s, \vec{\pi}) - \mathbb{P}^{\vec{\pi}}\{\sigma_1 > u\} e^{-\rho u} H(\vec{x}(u, \vec{\pi})) \\
&\quad + \mathbb{E}^{\vec{\pi}} \left[1_{\{\sigma_1 > u\}} e^{-\rho u} \mathbb{E}^{\vec{\Pi}_u} \left[\int_0^{t-u} e^{-\rho t} C(\vec{\Pi}_t) dt + 1_{\{\sigma_1 > t-u\}} e^{-\rho(t-u)} H(\vec{\Pi}_{t-u}) \right. \right. \\
&\quad \left. \left. + 1_{\{\sigma_1 \leq (t-u)\}} e^{-\rho \sigma_1} w(s - u - \sigma_1, \vec{\Pi}_{\sigma_1}) \right] \right] \\
&= Jw(u, s, \vec{\pi}) - \mathbb{P}^{\vec{\pi}}\{\sigma_1 > u\} e^{-\rho u} H(\vec{x}(u, \vec{\pi})) + \mathbb{E}^{\vec{\pi}} \left[1_{\{\sigma_1 > u\}} \cdot e^{-\rho u} Jw(t - u, s - u, \vec{\Pi}_u) \right] \\
&= Jw(u, s, \vec{\pi}) + \mathbb{P}^{\vec{\pi}}\{\sigma_1 > u\} e^{-\rho u} [Jw(t - u, s - u, \vec{x}(u, \vec{\pi})) - H(\vec{x}(u, \vec{\pi}))].
\end{aligned}$$

□

Proof of Lemma 4.1. Let $\vec{e}_i \in D$ denote the point whose i 'th component is equal to 1. To establish the result it is sufficient to find a closed ball with strictly positive radius around \vec{e}_i (e.g., a region of the form $\{\vec{\pi} \in D : \|\vec{\pi} - \vec{e}_i\| \leq \delta\}$ for some $\delta > 0$, where $\|\cdot\|$ denotes the Euclidian norm on \mathbb{R}^n) such that $H(\vec{\pi}) < v_1(s, \vec{\pi}) \leq V(s, \vec{\pi})$ for all points on this closed ball.

We first note that there exists a closed ball B_0 around \vec{e}_i with positive radius such that $H(\vec{\pi}) = \max_{k \in \mathcal{A}^*(i)} H_k(\vec{\pi})$, for $\vec{\pi} \in B_0$. Then on B_0 and for small $s > 0$ we have $v_1(s, \vec{\pi}) = \sup_{t \leq s} J_0 H(t, s, \vec{\pi}) = \max_{k \in \mathcal{A}^*(i)} \sup_{t \in [0, s]} J_0^{(k)} H(t, \vec{\pi})$, where $J_0^{(k)} H(t, \vec{\pi}) \triangleq$

$$\mathbb{E}^{\vec{\pi}} \left[e^{-I(t)} \right] e^{-\rho t} H_k(\vec{x}(t, \vec{\pi})) + \int_0^t e^{-\rho u} \sum_{j \in E} m_j(u, \vec{\pi}) (C(\vec{x}(u, \vec{\pi})) + \lambda_j S_j H(\vec{x}(u, \vec{\pi}))) du.$$

Then, using (2.14) we have $dJ_0^{(k)} H(t, \vec{\pi})/dt|_{t=0} =$

$$\begin{aligned}
&\left(-\rho - \sum_{j \in E} \lambda_j \pi_j \right) H_k(\vec{\pi}) + \sum_{j \in E} \mu_{k,j} \left(\sum_{l \in E} q_{l,j} \pi_l - \lambda_j \pi_j + \pi_j \sum_{l \in E} \lambda_l \pi_l \right) + C(\vec{\pi}) + \sum_{j \in E} \lambda_j \pi_j S_j H(\vec{\pi}) \geq \\
&\left(-\rho - \sum_{j \in E} \lambda_j \pi_j \right) H_k(\vec{\pi}) + \sum_{j \in E} \mu_{k,j} \left(\sum_{l \in E} q_{l,j} \pi_l - \lambda_j \pi_j + \pi_j \sum_{l \in E} \lambda_l \pi_l \right) + C(\vec{\pi}) + \sum_{j \in E} \lambda_j \pi_j S_j H_k(\vec{\pi}).
\end{aligned}$$

The right hand side of the inequality above is uniformly continuous on the compact set D . Its value at the point \vec{e}_i equals $c_i - \rho \mu_{k,i} + \sum_{j \neq i} (\mu_{k,j} - \mu_{k,i}) q_{k,j} > 0$. Hence for some $\delta_k > 0$ there exists an open ball (contained in B_0) with radius δ_k around \vec{e}_i such that $dJ_0^{(k)} H(t, \vec{\pi})/dt|_{t=0} > 0$ for all the points in this ball. Let B_k be the closed ball around the same point \vec{e}_i with radius $\delta_k/2$. Then on the intersection set $\bigcap_{k \in \mathcal{A}^*(i)} B_k$ the mapping $\vec{\pi} \mapsto dJ_0^{(k)} H(t, \vec{\pi})/dt|_{t=0}$ is strictly positive

and $\sup_{t \geq 0} J_0^{(k)} H(t, \vec{\pi}) > H_k(t, \vec{\pi})$ for all $k \in \mathcal{A}^*(i)$. This implies that $v_1(s, \vec{\pi}) > H(\vec{\pi})$ for all $s > 0$ on $\bigcap_{k \in \mathcal{A}^*(i)} B_k$. \square

Proof of Lemma 4.2. Let $i \in I^*$ for I^* defined in (4.14). To establish the result, we will find $\pi_i^s < 1$ such that $H(\vec{\pi}) = J_0 w(s, \vec{\pi})$ on $\{(s, \vec{\pi}) \in [0, T] \times D : \pi_i^s \leq \pi_i < 1\}$ for a bounded function $w(\cdot) \leq \|H\| = \bar{\mu} \triangleq \max_{i,k} \mu_{i,k}$. Since V is bounded by the same upper bound (recall that $c_i \leq 0$ for $i \in E$ by assumption) and satisfies $V(s, \vec{\pi}) = J_0 V(s, \vec{\pi})$ we will have $H(\cdot) = V(\cdot)$ on this region.

Part I: Let us first define

$$(A2.10) \quad F_k(t, \vec{\pi}) \triangleq \mathbb{E}^{\vec{\pi}} \left[e^{-I(t) - \rho t} \right] H_k(\vec{x}(t, \vec{\pi})) + \int_0^t e^{-\rho u} \sum_{j \in E} m_j(u, \vec{\pi}) \left[C(\vec{x}(t, \vec{\pi})) + \lambda_j \bar{\mu} \right] du.$$

Since $H(\vec{\pi}) \leq J_0 w(s, \vec{\pi}) = \sup_{t \in [0, s]} J w(t, s, \vec{\pi}) \leq \sup_{t \in [0, s]} \max_{k \in A} F_k(t, \vec{\pi}) = \max_{k \in A} \sup_{t \in [0, s]} F_k(t, \vec{\pi})$ (see (3.7)), it is enough to show that for some $\pi_i^s < 1$ we have $\sup_{t \geq 0} F_k(t, \vec{\pi}) = H_k(\vec{\pi})$ for all $k \in A$.

Let $\hat{\pi}_i < 1$ be a value such that $H(\vec{\pi}) = \max_{k \in \mathcal{A}^*} h_k(\vec{\pi})$, where $\mathcal{A}^* \triangleq \{k \in \mathcal{A} : \mu_{k,i} = \bar{\mu}\}$. That is, we have $\mu_{k,i} = \bar{\mu}$ for all $k \in \mathcal{A}^*$ (and $i \in I^*$). Note that $\hat{\pi}_i$ can for instance be selected as

$$\hat{\pi}_i = \max_{k \notin \mathcal{A}^*} \frac{\bar{\mu} - \min_{k,j} \mu_{k,j}}{2\bar{\mu} - \min_{k,j} \mu_{k,j} - a_{k,i}}.$$

Let us then define the hitting time $T(\vec{\pi}, \hat{\pi}_i) \triangleq \inf \{t \geq 0 : x_i(t, \vec{\pi}) \leq \hat{\pi}_i\}$. For $t \leq T(\vec{\pi}, \hat{\pi}_i)$, we have $\max_{k \in A} H_k(\vec{x}(t, \vec{\pi})) = \max_{k \in \mathcal{A}^*} H_k(\vec{x}(t, \vec{\pi}))$, which implies $\max_{k \in A} F_k(t, \vec{\pi}) = \max_{k \in \mathcal{A}^*} F_k(t, \vec{\pi})$. Note that we have

$$(A2.11) \quad \frac{dF_k(t, \vec{\pi})}{dt} = \sum_{i \in E} \mathbb{E}^{\vec{\pi}} \left[1_{\{M_t=i\}} e^{-I(t) - \rho t} \right] \left\{ -(\lambda_i + \rho) \cdot H_k(\vec{x}(t, \vec{\pi})) + \frac{dH_k(\vec{x}(t, \vec{\pi}))}{dt} + C(\vec{x}(t, \vec{\pi})) + \lambda_i \|H\| \right\}$$

where

$$(A2.12) \quad \frac{dH_k(\vec{x}(t, \vec{\pi}))}{dt} = \sum_{i \in E} \mu_{k,i} \left(\sum_j^n q_{ji} x_j(t, \vec{\pi}) - \lambda_i x_i(t, \vec{\pi}) + x_i(t, \vec{\pi}) \sum_j^n \lambda_j x_j(t, \vec{\pi}) \right)$$

due to (2.14). Let us denote $\underline{\mu} \triangleq \min_{k,i} \mu_{k,i}$. For $k \in A^*$, we have $H_k(\vec{x}(t, \vec{\pi})) = \bar{\mu} x_i(t, \vec{\pi}) + \sum_{i \neq i} \mu_{k,i} x_i(t, \vec{\pi}) \geq \bar{\mu} x_i(t, \vec{\pi}) + \underline{\mu} (1 - x_i(t, \vec{\pi}))$. Using this inequality, we get an upper bound for the derivative in (A2.11) as

$$(A2.13) \quad \frac{dF_k(t, \vec{\pi})}{dt} \leq \mathbb{E}^{\vec{\pi}} \left[e^{-I(t) - \rho t} \right] \left\{ \left(\bar{\lambda}(\bar{\mu} - \underline{\mu}) - \rho \underline{\mu} \right) (1 - x_i(t, \vec{\pi})) - \rho \bar{\mu} x_i(t, \vec{\pi}) + c_i x_i(t, \vec{\pi}) + \frac{dH_k(\vec{x}(t, \vec{\pi}))}{dt} \right\},$$

where $\bar{\lambda} \triangleq \max_{i \in E} \lambda_i$. Moreover, using (A2.12) it can be shown that for $k \in \mathcal{A}^*$ we have

$$\begin{aligned}
 (A2.14) \quad \frac{dH_k(\vec{x}(t, \vec{\pi}))}{dt} &= \sum_{j \in A} x_j(t, \vec{\pi}) \sum_{l \in E} \mu_{k,l} q_{lj} - \sum_{l \in E} \mu_{k,l} \lambda_l x_l(t, \vec{\pi}) + \sum_{l \in E} \mu_{k,l} x_l(t, \vec{\pi}) \sum_{j \in A} \lambda_j x_j(t, \vec{\pi}) \\
 &\leq n\bar{\mu} \left(\max_{l,j} |q_{lj}| \right) \left(1 - x_i(t, \vec{\pi}) \right) - \sum_{l \neq i} \mu_{k,l} \lambda_l x_l(t, \vec{\pi}) + \bar{\mu} x_i(t, \vec{\pi}) \sum_{j \neq k} \lambda_j x_j(t, \vec{\pi}) \\
 &\quad + \lambda_i x_i(t, \vec{\pi}) \sum_{l \neq i} \mu_{k,l} x_l(t, \vec{\pi}) + \left(\sum_{l \neq i} \mu_{k,l} x_l(t, \vec{\pi}) \right) \left(\sum_{j \neq k} \lambda_j x_j(t, \vec{\pi}) \right) \\
 &\leq \left(1 - x_i(t, \vec{\pi}) \right) \cdot \left(3 \cdot \bar{\mu} \cdot \bar{\lambda} + n \cdot \left(\max_{l,j} |q_{lj}| \right) \cdot \bar{\mu} \right)
 \end{aligned}$$

where the second line follows from the inequality $\sum_{l \in E} \mu_{k,l} q_{ll} \leq 0$ (recall that $\bar{\mu} = \mu_{k,i} = \max_{k,l} \mu_{k,l}$ and $q_{ii} = -\sum_{i \neq j} q_{ji}$). The equations (A2.13) and (A2.14) then imply that for $t < T(\vec{\pi}, \hat{\pi}_i)$, and for $k \in \mathcal{A}^*$,

$$(A2.15) \quad \frac{dF_k(t, \vec{\pi})}{dt} \leq \mathbb{E}^{\vec{\pi}} \left[e^{-I(t) - \rho t} \right] \cdot \left\{ -\rho \bar{\mu} x_i(t, \vec{\pi}) + c_i x_i(t, \vec{\pi}) + \left(1 - x_i(t, \vec{\pi}) \right) \cdot G \right\}.$$

where $G \triangleq 4 \cdot \bar{\mu} \cdot \bar{\lambda} + n \cdot (\max_{l,j} |q_{lj}|) \cdot \bar{\mu} - (\rho + \bar{\lambda}) \cdot \underline{\mu}$. Note that the assumption ' $\rho > 0$ or $c_i > 0$ ' in Lemma 4.2 assures that $dF_k(t, v\pi)/dt|_{t=0}$ is negative as $\pi_i \rightarrow 1$. Therefore, if we define

$$\hat{\pi}_i \triangleq \max \left\{ \hat{\pi}_i, \frac{G}{\rho \bar{\mu} - c_i + G} \right\} = \max \left\{ \hat{\pi}_i, \frac{4\bar{\mu}\bar{\lambda} + n(\max_{l,j} |q_{lj}|) \bar{\mu} - (\rho + \bar{\lambda}) \underline{\mu}}{-c_i + n\bar{\mu}(\max_{l,j} |q_{lj}|) + 3\bar{\lambda}\bar{\mu} + (\bar{\mu} - \underline{\mu})(\rho + \bar{\lambda})} \right\} < 1,$$

we have $dF_k(t, \vec{\pi})/dt \leq 0$ on $t \in [0, T(\vec{\pi}, \hat{\pi}_i)]$ for all $k \in \mathcal{A}^*$ and for all $\vec{\pi}$ such that $\pi_i > \hat{\pi}_i$. This implies that $JH(t, s, \vec{\pi}) \leq H(\vec{\pi})$ on this region.

Part II: Next, let $T(\vec{\pi}, \hat{\pi}_i)$ be the hitting time of the deterministic path $x_i(t, \vec{\pi})$ to the level $\hat{\pi}_i$. Below we show that there exists π_i^s such that

$$(A2.16) \quad F_k(t, \vec{\pi}) \leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{t \wedge \sigma_1} e^{-\rho t} c dt + e^{-\rho t \wedge \sigma_1} \bar{\mu} \right] \leq \bar{\mu} \pi_i^s + m(1 - \pi_i^s) \leq H(\vec{\pi})$$

for all $k \in \mathcal{A}$ (not just \mathcal{A}^*) and for all $t \geq T(\vec{\pi}, \hat{\pi}_i)$ on the region $\{\vec{\pi} \in D; \pi_i \geq \pi_i^s\}$. This will further imply that $JH(t, s, \vec{\pi}) \leq H(\vec{\pi})$ for all $t \geq 0$ for a point $\vec{\pi}$ falling on the latter region, and we will have $H(\vec{\pi}) \leq J_0 H(s, \vec{\pi}) = \sup_{t \in [0, s]} JH(t, s, \vec{\pi}) \leq H(\vec{\pi})$.

Note that the first inequality in (A2.16) follows from $C(\cdot) \leq c$ and $H(\cdot) \leq \bar{\mu}$. For a given value π_i^s the last inequality is true for all the points on $\{\vec{\pi} \in D; \pi_i \geq \pi_i^s\}$ since

$$H(\vec{\pi}) = \sup_{k \in \mathcal{A}^*} H_k(\vec{\pi}) = \bar{\mu} \pi_i + \sup_{k \in \mathcal{A}^*} \sum_{i \neq j} \mu_{k,i} \pi_i \geq \bar{\mu} \pi_i + m(1 - \pi_i) \geq \bar{\mu} \pi_i^s + m(1 - \pi_i^s).$$

Hence it remains to show that the second inequality holds for some π_i^s .

For $\pi_i > \hat{\pi}_i$ we have $\hat{\pi}_i = \pi_i + \int_0^{T(\vec{\pi}, \hat{\pi}_i)} \frac{d(x_i(t, \vec{\pi}))}{dt} dt$. Then, thanks to (2.14) we get $0 \geq \hat{\pi}_i - \pi_i =$

$$\int_0^{T(\vec{\pi}, \hat{\pi}_i)} \left(\sum_{j \in E} q_{ji} x_j(t, \vec{\pi}) - \lambda_i x_i(t, \vec{\pi}) + x_i(t, \vec{\pi}) \sum_{j \in E} \lambda_j x_j(t, \vec{\pi}) \right) dt \geq \int_0^{T(\vec{\pi}, \hat{\pi}_i)} (q_{ii} - \lambda_i) dt$$

$= (q_{ii} - \lambda_i) \cdot T(\vec{\pi}, \hat{\pi}_i)$, which further implies

$$(A2.17) \quad T(\vec{\pi}, \hat{\pi}_i) \geq (\pi_i - \hat{\pi}_i)/(-q_{ii} - \lambda_i).$$

Case I: $\rho > 0$. By (A2.17) we get the inequality $\mathbb{E}^{\vec{\pi}} \exp(-\rho \cdot T(\vec{\pi}, \hat{\pi}_i) \wedge \sigma_1) \leq$

$$\mathbb{E}^{\vec{\pi}} \exp\left(-\rho \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1\right]\right) = \int_0^\infty \exp\left(-\rho \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge u\right]\right) \sum_{i \in E} \mathbb{E}^{\vec{\pi}} \left[1_{\{M_u=i\}} e^{-I(u)} \lambda_i\right] du.$$

The last expression above is strictly decreasing in π_i and equals 1 at $\pi_i = \hat{\pi}_i$. Moreover the mapping $\pi_i \mapsto \bar{\mu}\pi_i + \underline{\mu}(1 - \pi_i)$ is increasing and equals $\bar{\mu}$ at $\pi_i = 1$. Therefore there exists a unique $\pi_i^s \in [\hat{\pi}_i, 1)$ defined as

$$(A2.18) \quad \pi_i^s \triangleq \inf \left\{ \pi_i \geq \hat{\pi}_i : \bar{\mu} \mathbb{E}^{\vec{\pi}} \exp\left(-\rho \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1\right]\right) \leq \bar{\mu}\pi_i + \underline{\mu}(1 - \pi_i) \right\} < 1,$$

such that the inequality in (A2.18) holds for all $\pi_i \in [\pi_i^s, 1]$. The definition of π_i^s implies that for all the points $\vec{\pi}$ with $\pi_i \geq \pi_i^s$ and for $t \geq T(\vec{\pi}, \hat{\pi}_i)$ we have

$$\begin{aligned} \mathbb{E} \left[\int_0^{t \wedge \sigma_1} e^{-\rho t} c dt + e^{-\rho t \wedge \sigma_1} \bar{\mu} \right] &\leq \bar{\mu} \mathbb{E} \left[e^{-\rho T(\vec{\pi}, \hat{\pi}_i) \wedge \sigma_1} \right] = \bar{\mu} \mathbb{E} \left[e^{-\rho T(\vec{\pi}, \hat{\pi}_i) \wedge \sigma_1} \right] \\ &\leq \bar{\mu} \mathbb{E}^{\vec{\pi}} \exp\left(-\rho \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1\right]\right) \leq \bar{\mu}\pi_i + \underline{\mu}(1 - \pi_i) \leq H(\vec{\pi}). \end{aligned}$$

This establishes (A2.16) and concludes the proof when $\rho > 0$.

Case II: $c > 0$. If $\rho > 0$, arguments given for Case I still holds. Hence we assume that $\rho = 0$. Using (A2.17) again, we obtain

$$\mathbb{E}^{\vec{\pi}} \left[T(\vec{\pi}, \hat{\pi}_i) \wedge \sigma_1 \right] \geq \mathbb{E}^{\vec{\pi}} \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1 \right] = \int_0^\infty \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge u \right] \sum_{j \in E} \lambda_j m_j(u, \vec{\pi}) du.$$

The last expression above equals to 0 at $\pi_i = \hat{\pi}_i$ and it is strictly increasing in π_i for $\pi_i \geq \hat{\pi}_i$. Therefore there exists a unique point

$$\pi_i^s \triangleq \inf \left\{ \pi_i \geq \hat{\pi}_i : -c \mathbb{E}^{\vec{\pi}} \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1 \right] + \bar{\mu} \leq \bar{\mu}\pi_i + \underline{\mu}(1 - \pi_i) \right\} < 1,$$

Then for the points $\vec{\pi}$ with $\pi_i \geq \pi_i^s$ and for $t \geq T(\vec{\pi}, \hat{\pi}_i)$ we have

$$\begin{aligned} \mathbb{E} \left[\int_0^{t \wedge \sigma_1} c dt + \bar{\mu} \right] &= c \mathbb{E} [t \wedge \sigma_1] + \bar{\mu} \leq c \mathbb{E} \left[T(\vec{\pi}, \hat{\pi}_i) \wedge \sigma_1 \right] + \bar{\mu} \\ &\leq c \mathbb{E}^{\vec{\pi}} \left[\frac{\pi_i - \hat{\pi}_i}{-q_{ii} + \lambda_i} \wedge \sigma_1 \right] + \bar{\mu} \leq \bar{\mu}\pi_i + \underline{\mu}(1 - \pi_i) \leq H(\vec{\pi}), \end{aligned}$$

and this concludes the proof. \square

Proof of Lemma 4.3. The first inequality in (4.18) is obvious. To show the second inequality let τ be an \mathbb{F} -stopping time. Then, we have

$$(A2.19) \quad \begin{aligned} \mathbb{E}^{\vec{\pi}} \left[\int_0^\tau e^{-\rho t} k(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) \right] &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{\tau \wedge T} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau \wedge T} H(\vec{\Pi}_{\tau \wedge T}) \right] \\ &\quad + \mathbb{E}^{\vec{\pi}} \left[1_{\{\tau \geq T\}} \left(\int_T^\tau e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho \tau} H(\vec{\Pi}_\tau) - e^{-\rho T} H(\vec{\Pi}_T) \right) \right]. \end{aligned}$$

If $\rho > 0$, the last expectation above is bounded above by $e^{-\rho T}(\|C\| + 2 \cdot \|H\|)$. Then taking the supremum over all τ 's on both sides we obtain (4.18).

On the other hand, if $\rho = 0$ and $\max_{i \in E} c_i < 0$, we may safely restrict ourselves to the set of stopping times τ for which $\mathbb{E}[\tau] \leq (\min_{k,i} \mu_{k,i} - \max_{k,i} \mu_{k,i}) / \max_{i \in E} c_i$: the expected reward associated with any stopping time having a higher expected value is dominated by the reward achieved upon stopping immediately. Then, the second expectation in (A2.19) is bounded above by

$$2 \cdot \|H\| \cdot \mathbb{P}\{\tau > T\} \leq 2 \cdot \|H\| \frac{\mathbb{E}[\tau]}{T} \leq \frac{2 \cdot \|H\|}{T} \frac{(\min_{k,i} \mu_{k,i} - \max_{k,i} \mu_{k,i})}{\max_{i \in E} c_i},$$

thanks to Markov's inequality. Then, the inequality in (4.18) follows after taking the supremums over τ again. \square

Proof of (4.21). Let $U_\varepsilon^{(m)}$ denote the stopping rule in (4.20) for notational convenience. Since $U_\varepsilon^{(m)} \wedge T \leq U_\varepsilon^{(m)} \leq U_0(\infty, \vec{\pi})$, the arguments of [12, Proposition 3.11 and Section 4.1] give

$$V(T, \vec{\pi}) \leq V(\infty, \vec{\pi}) = \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon^{(m)} \wedge T} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho(U_\varepsilon^{(m)} \wedge T)} V(\infty, \vec{\Pi}_{U_\varepsilon^{(m)} \wedge T}) \right].$$

On the event $\{U_\varepsilon^{(m)} \leq T\}$, we use the inequality $V(\infty, \vec{\Pi}_{U_\varepsilon^{(m)}}) - \varepsilon - \text{Err}_\infty(m) \leq H(\vec{\Pi}_{U_\varepsilon^{(m)}})$, $\mathbb{P}^{\vec{\pi}}$ -a.s., to obtain

$$\begin{aligned} V(T, \vec{\pi}) &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon^{(m)} \wedge T} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho(U_\varepsilon^{(m)} \wedge T)} H(\vec{\Pi}_{U_\varepsilon^{(m)} \wedge T}) + \varepsilon + \text{Err}_\infty(m) \right. \\ &\quad \left. + 1_{\{U_\varepsilon^{(m)}(\infty, \vec{\pi}) > T\}} e^{-\rho T} [V(\infty, \vec{\Pi}_T) - H(\vec{\Pi}_{U_\varepsilon^{(m)}})] \right] \\ &\leq \mathbb{E}^{\vec{\pi}} \left[\int_0^{U_\varepsilon^{(m)} \wedge T} e^{-\rho t} C(\vec{\Pi}_t) dt + e^{-\rho(U_\varepsilon^{(m)} \wedge T)} H(\vec{\Pi}_{U_\varepsilon^{(m)} \wedge T}) \right] \\ &\quad + \varepsilon + \text{Err}_\infty(m) + e^{-\rho T} \text{Err}_\infty(0) \mathbb{P}\{U_\varepsilon^{(m)} > T\}. \end{aligned}$$

If $\rho > 0$, we obtain (4.21) by removing the last probability. Otherwise we can use Markov's inequality $\mathbb{P}\{U_\varepsilon^{(m)} > T\} \leq \mathbb{E}[U_\varepsilon^{(m)}]/T \leq \mathbb{E}[U_\varepsilon^0]/T \leq \max_{k,i} \mu_{k,i}/[(\min_{i \in E} c_i)T]$ as in the proof of Lemma 4.18, and (4.21) follows. \square

Proof of Remark 6.1. The first claim on immediate stopping if $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) \leq \mu_{2,1} + \mu_{1,2}$ is an immediate corollary of [32, Theorem 2.1].

Let us now assume that $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) > \mu_{2,1} + \mu_{1,2}$. For the problem with two hypotheses, we have $H(\vec{\pi}) = \min\{\mu_{1,2}\pi_2; \mu_{2,1}\pi_1\}$, and recall that $v_1(T, \vec{\pi}) = \inf_{t \in [0, s]} JH(t, \vec{\pi})$. For $\vec{\pi} = (\pi_1, \pi_2)$ with $\pi_2 \in (\lambda_1\mu_{2,1}/(\lambda_2\mu_{1,2} + \lambda_1\mu_{2,1}), \mu_{2,1}/(\mu_{2,1} + \mu_{1,2}))$ and for small $t > 0$, evaluating the expression $JH(t, \vec{\pi})$ gives

$$\left[\pi_1 e^{-\lambda_1 t} + \pi_2 e^{-\lambda_2 t} \right] \mu_{1,2} x_2(t, \vec{\pi}) + \int_0^t \sum_{j=1}^2 \pi_i e^{-\lambda_i u} \left(1 + \lambda_j \left(\mu_{2,1} \frac{\lambda_1 x_1(u, \vec{\pi})}{\lambda_1 x_1(u, \vec{\pi}) + \lambda_2 x_2(u, \vec{\pi})} \right) \right) du,$$

and using the dynamics of $t \mapsto \vec{x}(t, \vec{\pi})$ in (2.14) we obtain

$$(A2.20) \quad \frac{dJH(t, \vec{\pi})}{dt} = [1 + \mu_{2,1}\lambda_1] \cdot \pi_1 e^{-\lambda_1 t} + [1 - \mu_{1,2}\lambda_2] \cdot \pi_2 e^{-\lambda_2 t}.$$

With $t = 0$ and $\vec{\pi} = (\mu_{1,2}/(\mu_{2,1} + \mu_{1,2}) + \delta, \mu_{2,1}/(\mu_{2,1} + \mu_{1,2}) - \delta)$, for $\delta > 0$ small, the derivative becomes

$$\left. \frac{dJH(t, \vec{\pi})}{dt} \right|_{t=0, \vec{\pi}=(\cdot, \cdot)} = \frac{1}{\mu_{2,1} + \mu_{1,2}} [\mu_{2,1} + \mu_{1,2} + \mu_{2,1}\mu_{1,2}(\lambda_1 - \lambda_2)] + \delta(\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2).$$

Under the assumption $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) > \mu_{2,1} + \mu_{1,2}$, the last expression is negative for δ sufficiently small. This implies that $v_1(T, \vec{\pi}) < H(\vec{\pi})$ for small values of $T > 0$ at points $\vec{\pi}$, for which $\pi_2 = \mu_{2,1}/(\mu_{2,1} + \mu_{1,2}) - \delta$ where

$$\delta < \frac{\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) - \mu_{2,1} - \mu_{1,2}}{(\mu_{2,1} + \mu_{1,2})(\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2)}.$$

Since $b_1(0) = \mu_{2,1}/(\mu_{2,1} + \mu_{1,2})$, it follows that the boundary curve $T \mapsto b_1(T)$ is discontinuous at $T = 0$ (see the lower curve in Figure 3).

The expression in (A2.20) with $t = 0$ indicates that $dJH(t, \vec{\pi})/dt|_{t=0}$ is decreasing in π_2 and vanishes at the point $\vec{\pi}$ with

$$\pi_2 = \frac{1 + \mu_{2,1}\lambda_1}{\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2} \leq \frac{\mu_{2,1}}{\mu_{2,1} + \mu_{1,2}},$$

where the inequality is due to the assumption $\mu_{2,1}\mu_{1,2}(\lambda_2 - \lambda_1) > \mu_{2,1} + \mu_{1,2}$. This implies that

$$\left\{ (T, \vec{\pi}) : \pi_2 \leq \frac{\mu_{2,1}}{\mu_{2,1} + \mu_{1,2}} \text{ and } V_1(T, \vec{\pi}) = H(\vec{\pi}) \right\} \subseteq \left\{ (T, \vec{\pi}) : \pi_2 \leq \frac{1 + \mu_{2,1}\lambda_1}{\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2} \right\}$$

At the point $\vec{\pi}$ with $\pi_2 = (1 + \mu_{2,1}\lambda_1)/(\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2)$ the expression for $dJH(t, \vec{\pi})/dt$ in (A2.20) is strictly positive for small $t > 0$. Then, we can find a value of $u > 0$ such that

$$v_1(T, \vec{\pi}) = H(\vec{\pi}), \quad \text{for } \vec{\pi} = \left(\frac{\mu_{1,2}\lambda_2}{\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2}, \frac{1 + \mu_{2,1}\lambda_1}{\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2} \right) \text{ and } T \in [0, u].$$

This further implies

$$v_1(T, \vec{\pi}) = H(\vec{\pi}) \quad \text{on} \quad \left\{ (T, \vec{\pi}) : T \in [0, u] \text{ and } \pi_2 \leq \frac{1 + \mu_{2,1}\lambda_1}{\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2} \right\},$$

since the region $\{\vec{\pi} \in D : V(T, \vec{\pi}) = H(\vec{\pi})\}$ is convex for each T (see Remark 4.3), and we have $v_1(T, \vec{\pi}) = H(\vec{\pi})$, for all $T > 0$ at $\vec{\pi} = (1, 0)$. Recall that the deterministic part $t \mapsto \vec{x}(t, \pi)$ drifts towards the point $(1, 0)$. Then, by induction we conclude that $v_n(T, \vec{\pi}) = H(\vec{\pi})$ for all $n \in \mathbb{N}$, which implies that $\lim_{n \rightarrow \infty} v_n(T, \vec{\pi}) = V(T, \vec{\pi}) = H(\vec{\pi})$ on the same region.

As a result, we see that if the solution of the problem is not trivial, the lower boundary curve $b_1(T)$ is discontinuous at $T = 0$, and there is an initial region over which the curve stays flat at level $\pi_2 = (1 + \mu_{2,1}\lambda_1)/(\mu_{2,1}\lambda_1 + \mu_{1,2}\lambda_2)$ as in Figure 3. \square

REFERENCES

- [1] E. ARJAS, P. HAARA, AND I. NORROS, *Filtering the histories of a partially observed marked point process*, Stochastic Processes and their Applications, 40 (1992), pp. 225–250.
- [2] J. A. BATHER, *An optimal stopping problem with costly information*, Bulletin of Institute for International Statistics, 45 (1973), pp. 9–24.
- [3] E. BAYRAKTAR, S. DAYANIK, AND I. KARATZAS, *Adaptive Poisson disorder problem*, Annals of Applied Probability, 16 (3) (2006), pp. 1190–1261.
- [4] A. BENSOUSSAN, *Stochastic control of partially observable systems*, Cambridge University Press, Cambridge, 1992.
- [5] D. P. BERTSEKAS, *Dynamic programming and stochastic control*, Academic Press, New York, 1976. Mathematics in Science and Engineering, 125.
- [6] P. BREMAUD, *Point Processes and Queues*, Springer, New York, 1981.
- [7] O. L. V. COSTA AND M. H. A. DAVIS, *Approximations for optimal stopping of a piecewise-deterministic process*, Mathematics of Control, Signals, and Systems, 1 (1988), pp. 123–146.

- [8] D. COX AND V. ISHAM, *Point Processes*, Chapman and Hall, London, 1980.
- [9] J. N. DARROCH AND K. W. MORRIS, *Passage-time generating functions for continuous-time finite Markov chains*, Journal of Applied Probability, 5 (1968), pp. 414–426.
- [10] M. H. A. DAVIS, *Markov Models and Optimization*, Chapman & Hall, London, 1993.
- [11] S. DAYANIK AND C. GOULDING, *Detection and identification of an unobservable change in the distribution of a Markov-modulated random sequence*, IEEE Transactions on Information Theory, 55 (7) (2009), pp. 3323–3345.
- [12] S. DAYANIK, V. POOR, AND S. SEZER, *Bayesian sequential multi-hypothesis testing for (compound) Poisson processes*, Stochastics, 80(1) (2008), pp. 19–50.
- [13] J.-P. DÉCAMPS, T. MARIOTTI, AND S. VILLENEUVE, *Investment timing under incomplete information*, Mathematics of Operations Research, 30 (2005), pp. 472–500.
- [14] R. J. ELLIOTT, L. AGGOUN, AND J. B. MOORE, *Hidden Markov models*, vol. 29 of Applications of Mathematics (New York), Springer-Verlag, New York, 1995. Estimation and control.
- [15] R. J. ELLIOTT AND W. P. MALCOLM, *General smoothing formulas for Markov-modulated Poisson observations*, Institute of Electrical and Electronics Engineers. Transactions on Automatic Control, 50 (2005), pp. 1123–1134.
- [16] E. A. FEINBERG, *Continuous-time discounted jump Markov decision processes: a discrete-event approach*, Mathematics of Operations Research, 29 (2004), pp. 492–524.
- [17] A. FRIEDMAN, *Optimal stopping for random evolution of multidimensional Poisson processes with partial information*, Friedman, A., and Pinsky, M., eds., Academic Press, New York, (1978).
- [18] C. FUH, *SPRT and CUSUM in hidden Markov models*, Annals of Statistics, 31 (2003), pp. 942–977.
- [19] P. V. GAPEEV, *Problems of sequential discrimination of hypotheses for a compound Poisson process with exponential jumps*, Upsekhi Mat. Nauk, 57 (2002), pp. 171–172.
- [20] U. S. GUGERLI, *Optimal stopping of a piecewise-deterministic Markov process*, Stochastics, 19 (1986), pp. 221–236.
- [21] R. JENSEN, *Adoption and diffusion of an innovation of uncertain profitability*, Journal of Economic Theory, 27 (1982), pp. 182–193.
- [22] U. JENSEN, *Monotone stopping rules for stochastic processes in a semimartingale representation with applications*, Optimization, 20 (1989), pp. 837–852.
- [23] U. JENSEN, *An optimal stopping problem in risk theory*, Scandinavian Actuarial Journal, 2 (1997), pp. 149–159.
- [24] U. JENSEN AND G.-H. HSU, *Optimal stopping by means of point process observations with applications in reliability*, Mathematics of Operations Research, 18 (1993), pp. 645–657.
- [25] S. KARLIN AND H. TAYLOR, *An Introduction to stochastic modelling*, Academic Press, third ed., 1998.
- [26] S. LENHART AND Y. C. LIAO, *Integro-differential equations associated with optimal stopping time of a piecewise-deterministic process*, Stochastics, 15 (1985), pp. 183–207.
- [27] R. S. LIPTSER AND A. N. SHIRYAEV, *Statistics of Random Processes, I and II*, Springer-Verlag, Berlin, 2001.
- [28] W. S. LOVEJOY, *A survey of algorithmic methods for partially observed Markov decision processes*, Annals of Operations Research, 28 (1991), pp. 47–66.
- [29] V. MAKIS AND X. JIANG, *Optimal replacement under partial observations*, Mathematics of Operations Research, 28 (2003), pp. 382–394.
- [30] G. MAZZIOTTO, *Approximations of the optimal stopping problem in partial observation*, Journal of Applied Probability, 23 (1986), pp. 341–354.
- [31] K. F. MCCARDLE, *Information acquisition and the adoption of new technology*, Management Science, 31 (1985), pp. 1372–1389.
- [32] G. PESKIR AND A. N. SHIRYAEV, *Sequential testing problems for Poisson processes*, Annals of Statistics, 28 (2000), pp. 837–859.
- [33] ———, *Optimal Stopping and Free-boundary problems*, Birkhauser-Verlag, Lectures in Mathematics, ETH Zurich, 2006.
- [34] A. SCHÖTTL, *Optimal stopping of a risk reserve process with interest and cost rates*, Journal of Applied Probability, 35 (1998), pp. 115–123.
- [35] A. N. SHIRYAEV, *Optimal stopping rules*, Springer-Verlag, Berlin, 1978.
- [36] R. SMALLWOOD AND E. SONDIK, *The optimal control of partially observable Markov processes over a finite horizon*, Operations Research, 21 (1973), pp. 1071–1088.
- [37] W. STADJE, *Maximal wearing-out of a deteriorating system: An optimal stopping approach*, European Journal of Operational Research, 73 (1994), pp. 472–479.
- [38] ———, *An optimal stopping problem with two levels of incomplete information*, Mathematical Methods of Operations Research, 45 (1997), pp. 119–131.
- [39] C. ULU AND J. E. SMITH, *Information acquisition and technology adoption*, 2007. To appear in Operations Research.

(M. Ludkovski) DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY, UNIVERSITY OF CALIFORNIA SANTA BARBARA, CA 93106-3110

E-mail address: ludkovski@pstat.ucsb.edu

(S. O. Sezer) SCHOOL OF ENGINEERING AND APPLIED SCIENCES, SABANCI UNIVERSITY, ISTANBUL

E-mail address: sezer@sabanciuniv.edu